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The Indices of Permutations and the Derivation therefrom of Functions of a Single Variable Associated with the Permutations of any Assemblage of Objects.

BY MAJOR P. A. MACMAHON.

Table of Contents.

PAGE.

Introduction 282

§ 1.

Arts. 1–5. Definitions of the Greater, Equal and Lesser Indices of a Permutation and of the Classes Corresponding to Them . . . 283

Art. 6. Evaluation of Σx^p for the Permutations of Any Given Assemblage 285

Arts. 7, 8. Properties of Σx^p 289

Art. 9. Proof that $\Sigma x^r = \Sigma x^p$ 289

Art. 10. The Expression for Σx^{p+q} 289

Art. 11. Difference Equation for Σx^p 290

Art. 12. Remarks on Σx^p for the Assemblage $\alpha^i\beta^j$—Connection with Partitions 291

Art. 13. Equal Index Function for the Assemblage $\alpha^i\beta^j$ and Property of $p-r$ 292

Art. 14. Expression of Σx^{p+r} for the Assemblage $\alpha^i\beta^j$ 293

Art. 15. The Number $p-r$ for the Assemblage $\alpha^i\beta^j\gamma^k$ 293

Arts. 16, 17. The Parity of the Greater Index and Property in Respect of any Prime Modulus 293

Arts. 18, 19. Average Values of the Indices 294

Art. 20. Average Value of the Square of the Greater Index 296

Arts. 21–23. Average Values of Higher Powers of the Greater Index . . 298

Arts. 24–25. Average Value of $p_1^\nu + p_2^\nu + \dots + p_m^\nu$ where $P = p_1 + p_2 + \dots + p_m$ 299

Art. 26. Average Values for $\nu=2$ and $\nu=3$ 300

Art. 27. Average Value of the Class of a Permutation 301

Art. 28. Average Number of Equal Contacts in a Permutation 302

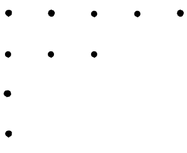
Art. 29. Average Values Connected with the Equal Index 302

§ 2.

PAGE.
Arts. 30, 31. The Major and Minor Indices P, R 303
Arts. 32, 33. Important Property of $P-R$ 304
Art. 34. Relation between Σx^P and Σx^R 306
Art. 35. Difference Equation for Σx^P 306
Arts. 36, 37. The Assemblage $\alpha^i\beta^j\gamma^k$. Proof that $\Sigma x^P = \Sigma x^R$ 307
Art. 38. Preliminary Note on Σx^P for the Assemblage $\alpha^i\beta^j\gamma^r$ 308
Art. 39. Derivation of Σx^R from Σx^P 308
Art. 40. Expression for Σx^{P+R} 309
Arts. 41–44. Average Values of the Major and Minor Indices. Average Values after Arts. 24–25 309
Art. 45. Proof that for the most General Assemblage $\Sigma R = \Sigma P$ 311
Arts. 46–48. Connection between Σx^P and Partitions 312
Arts. 49–55. Investigation of Σx^P for the Assemblage $\alpha^i\beta^j\gamma^k$ 314
Art. 56. Remarks upon Σx^P for the Assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$ when $s>3$. 320
Art. 57. Crude Expression for Σx^P in the Case of the Assemblage $\alpha\beta\gamma\delta$ 321

Introduction.

This paper arises out of the postscript to the "Memoir on the Theory of the Partitions of Numbers," Part V. *Phil. Trans. R. S.*, Vol. CCXI, A 473, January, 1911. In that paper I derived algebraic functions of a single variable from *certain* permutations of any assemblage of objects. The permutations were restricted to be such as fulfilled certain conditions, and by reason of their intimate association with lattices or gratings of nodes complete or incomplete they were termed lattice-permutations. The discovery of the functions led to the solution of the problem of two-demensional partition at the points of a Ferrers-Sylvester graph such as is figured below,



where we have a system of nodes constituting the graph of a partition of a number. In two-dimensional partition we have a *certain* generalization of ordinary or one-dimensional partition. In the latter the partition is constituted by numbers placed at the nodes of a single row in descending order of magnitude, the sum of the numbers being equal to the number which is partitioned. In the former numbers are placed at the nodes of the two-dimensional system in such wise that the numbers are in descending order of magnitude alike in *each row and in each column* of the system. The system of nodes is supposed to be given, and when the sum of the numbers placed at the nodes is equal to the partitioned number, the system of numbers in two dimensions is a two-dimensional partition of the given number *quâ* the given system of nodes. It will be observed that the system of nodes is itself defined by a one-dimensional partition of some number and the enumerative problem that was solved may be stated as the determination of the number of partitions of any number *quâ* a system of nodes defined by any one-dimensional partition of any number. The generalization that was made was brought to a conclusion in a subsequent paper, "Memoir on the Theory of the Partitions of Numbers," Part VI. *Phil. Trans. R. S.*, Vol. CCXI, A 479. It depends upon the relative position of the parts and may be made to depend upon linear Diophantine analysis. The problem is certainly very simply stated in Diophantine terms, but the actual solution reached was obtained from generating functions derived from *permutations* in association with the given system of nodes. The permutations were made to lead to algebraic functions through the agency of numbers derived from the permutations. In the postscript alluded to there was an adumbration of a similar theory connected with the entire set of permutations of any assemblage of objects. The present paper opens the discussion of this question and formally introduces various kinds of indices of permutations. So much of the matter that the postscript discussed as involved a limit to the magnitude of the parts is not broached here except in one place in an incidental manner. The theory of the greater and lesser indices is complete. That of the equal index is incomplete so far as the algebraic functions are concerned, but the theory of the average values is complete in this case also. As regards the theory of the major and minor indices, the discussion is complete when the assemblage only contains three *kinds* of objects and the *crude* form of generating function has been obtained for the most general assemblage. The theory of average values is complete in this case also. I hope, if permitted, to continue the discussion on a future occasion.

A communication on the subject-matter of this paper was made to the Mathematical Subsection of Section A of the British Association for the Advancement of Science at the Dundee meeting, September, 1912. P. A. M.

§ 1. *Indices of the First Genus.*

1. The assemblage of objects I take to be

$$\alpha^i\beta^j\gamma^k\ldots\ldots \quad \text{or} \quad \alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\ldots\ldots$$

indifferently.

A contact in a permutation of these letters, which is (say) $\alpha_s\alpha_t$, is called a major contact, an equal contact or a minor contact, according as $s >, =,$ or $< t$.

The first definitions have reference to the major contacts of a permutation.

2. *The Greater Index of a Permutation.* Of $\alpha^4\beta^2\gamma^3$ let any permutation be (say)

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma.$$

Whenever a letter is the left-hand member of a major contact, I write under it a number which shows how many places it is from the left of the permutation. Thus if the s-th letter stands before a letter prior to it in alphabetical (or numerical) order, I place under it the number s. In this way we obtain

$$\underset{1}{\beta}\alpha\alpha\alpha\gamma\underset{6}{\gamma}\underset{7}{\beta}\alpha\gamma.$$

We add up the numbers so placed and obtain $1+6+7=14$, a number which I call the "greater index" of the permutation. Similarly, if, in any permutation of any assemblage of letters or of ordered objects, the p_1-th, p_2-th, p_3-th, ... letters are the left-hand members of major contacts, we have the definition

$$\text{Greater index} = p_1+p_2+p_3+\ldots\ldots=p.$$

If, in so forming the number p, we have to add m numbers or, in other words, if the permutation possesses m major contacts, I speak of the permutation as being of "Class m" *quâ* major contacts.

3. *The Equal Index of a Permutation.* If, in any permutation, the q_1-th, q_2-th, q_3-th, ... letters immediately precede letters identical with themselves, so that such letters are the left-hand members of equal contacts, I make the definition

$$\text{Equal index} = q_1+q_2+q_3+\ldots\ldots=q.$$

Thus from

$$\beta\underset{2}{\alpha}\underset{3}{\alpha}\alpha\underset{5}{\gamma}\gamma\beta\alpha\gamma$$

we obtain the equal index $2+3+5=10$.

If, in so forming the number q, we have to add m numbers, or if, in other words, the permutation possesses m equal contacts, I speak of the permutation as being of "Class m" *quâ* equal contacts.

4. *The Lesser Index of a Permutation.* If, in any permutation, the r_1-th, r_2-th, r_3-th, ... letters immediately precede letters which are later in alphabetical order, so that such letters are the left-hand members of minor contacts, I make the definition

$$\text{Lesser index} = r_1+r_2+r_3+\ldots\ldots=r.$$

If in so forming the number r we have to add m numbers, or, in other words, if the permutation possesses m minor contacts, I speak of the permutation as being of "Class m" *quâ* minor-contacts.

5. *The Greater and Equal Index.* This refers to letters which immediately precede letters which are *not later* in alphabetical order and is equal to the sum of the greater and equal indices, or to $p+q$.

In the same manner we may have to consider other combinations of the indices p, q, r.

6. *Evaluation of Σx^p for Permutations of Any Given Assemblage.* In the case of the assemblage $\alpha^i\beta^j\gamma^k\ldots.$ I wish to give a formal proof of the result

$$\Sigma x^p=\frac{(1-x)(1-x^2)\ldots(1-x^{i+j+k+\ldots})}{(1-x)(1-x^2)\ldots(1-x^i)\cdot(1-x)(1-x^2)\ldots(1-x^j)\cdot(1-x)(1-x^2)\ldots(1-x^k)\cdot\ldots}$$

of which an indication only was given in the postscript to the paper, to which reference has been made in the preamble above.

In Cayley's notation I write

$$1-x^s=(\mathbf{s}),$$

so that the result to be established is written

$$\Sigma x^p=\frac{(1)(2)\ldots.(\mathbf{i}+\mathbf{j}+\mathbf{k}+\ldots.)}{(1)(2)\ldots.(\mathbf{i})\cdot(1)(2)\ldots.(\mathbf{j})\cdot(1)(2)\ldots.(\mathbf{k})\cdot\ldots.}.$$

Consider a system of rows of nodes, containing i, j, k, ... nodes respectively and at these nodes numbers to be placed in such wise that there is a descending order of magnitude in each row. The number zero is not excluded from being placed at a node and there is no condition whatever in regard to the numbers which appear in the columns of nodes. If it be asked,—In how many ways can numbers be so placed that their sum is w?—it is readily seen that the answer is supplied by the coefficient of x^w in the ascending expansion of the algebraic fraction

$$\frac{1}{(1)(2)\ldots.(\mathbf{i})\cdot(1)(2)\ldots.(\mathbf{j})\cdot(1)(2)\ldots.(\mathbf{k})\cdot\ldots.};$$

because if w_i, w_j, w_k, ... be any numbers such that $w_i+w_j+w_k+\ldots.=w$, the members w_i, w_j, w_k, ... can be partitioned in the 1st, 2d, 3d, ... rows in numbers of ways indicated by the generating functions

$$\frac{1}{(1)(2)\ldots.(\mathbf{i})},\quad\frac{1}{(1)(2)\ldots.(\mathbf{j})},\quad\frac{1}{(1)(2)\ldots.(\mathbf{k})},\quad\ldots.$$

respectively.

The present proof consists in showing that the same generating function can be exhibited as the quotient of Σx^p, p being the greater index of a permutation of $\alpha^i\beta^j\gamma^k\ldots.$, by the function

$$(1)(2)\ldots.(i+j+k+:\ldots).$$

Consider a simple particular case, $i=3$, $j=2$, to which belongs the system of nodes

• • •
• •

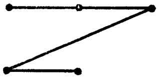
and suppose numbers to be placed at the nodes, viz.,

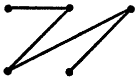
$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & \end{matrix}$$

in such wise that $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2$. Place also the quantities α, β of the assemblage $\alpha^3\beta^2$ at the nodes in the manner

$$\begin{matrix} \alpha & \alpha & \alpha \\ \beta & \beta & \end{matrix}$$

We are about to enquire into the number of ways of placing the numbers at the nodes by arranging them in all possible descending orders, and then enumerating each order separately. We must determine the lines of route through the system of nodes which may give a descending order of magnitude of the numbers a_1, a_2, a_3, b_1, b_2 collectively. These numbers may be in any permutation so long as the two orders a_1, a_2, a_3; b_1, b_2 are in evidence. Thus through the system of nodes we may have the lines of route


$\alpha\alpha\alpha\beta\beta$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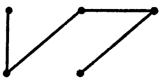
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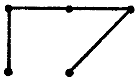
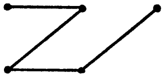

$\beta\alpha\alpha\alpha\beta$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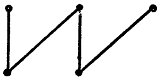
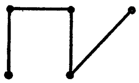
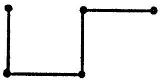
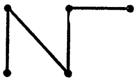
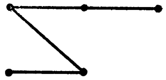

$\alpha\beta\alpha\beta\alpha$


$\beta\alpha\alpha\beta\alpha$


$\alpha\beta\beta\alpha\alpha$


$\beta\alpha\beta\alpha\alpha$


$\beta\beta\alpha\alpha\alpha$

Beneath each line of route is placed the corresponding permutation of $\alpha^3\beta^2$. There is obviously one line of route for each permutation.

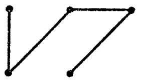
If we take any one of these lines of route, we may lay out five numbers in descending order of magnitude along it. In the general case each line of route is $i+j+k+\ldots.$ nodes long, and we may lay out $i+j+k+\ldots.$ numbers in descending order along the one selected. The number of ways in which this can be done is given by the coefficient of x^w in

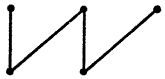
$$\{(1)(2)\ldots.(i+j+k+\ldots.)\}^{-1};$$

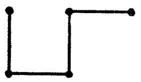
and if we so proceeded, in the case of each permutation of $\alpha^i\beta^j\gamma^k\ldots.$, we would obtain this coefficient multiplied by the number of permutations of $\alpha^i\beta^j\gamma^k\ldots..$ We would not, however, thus obtain the required generating function, because, in general, many systems of numbers may be laid out along more than one line of route. For example, the arrangement

$$\begin{matrix} 4 & 2 & 2 \\ 3 & 2 & \end{matrix}$$

occurs along each of the lines of route







For the enumeration before us we must arrange that each line of route is utilized without overlapping. This is accomplished by imposing certain conditions upon the successions of numbers which appertain to the various lines of route.

Observe the subjoined scheme of descending orders for the particular case under examination:

1.	$\alpha\alpha\alpha\beta\beta$,	$a_1 \geq a_2 \geq a_3 \geq b_1 \geq b_2$,
2.	$\alpha\alpha\beta\alpha\beta$,	$a_1 \geq a_2 \geq b_1 > a_3 \geq b_2$,
3.	$\alpha\beta\alpha\alpha\beta$,	$a_1 \geq b_1 > a_2 \geq a_3 \geq b_2$,
4.	$\beta\alpha\alpha\alpha\beta$,	$b_1 > a_1 \geq a_2 \geq a_3 \geq b_2$,
5.	$\alpha\alpha\beta\beta\alpha$,	$a_1 \geq a_2 \geq b_1 \geq b_2 > a_3$,
6.	$\alpha\beta\alpha\beta\alpha$,	$a_1 \geq b_1 > a_2 \geq b_2 > a_3$,
7.	$\beta\alpha\alpha\beta\alpha$,	$b_1 > a_1 \geq a_2 \geq b_2 > a_3$,
8.	$\alpha\beta\beta\alpha\alpha$,	$a_1 \geq b_1 \geq b_2 > a_2 \geq a_3$,
9.	$\beta\alpha\beta\alpha\alpha$,	$b_1 > a_1 \geq b_2 > a_2 \geq a_3$,
10.	$\beta\beta\alpha\alpha\alpha$,	$b_1 \geq b_2 > a_1 \geq a_2 \geq a_3$,

where the descending order on the right appertains to the permutation on the left and therefore to the line of route which the permutation denotes. Wherever in the permutation there is either an equal contact or a minor contact, the symbol $\geq$ is placed in the corresponding place in the descending order; but wherever there is a major contact, the symbol employed is $>$. The successions of numbers thus defined are all distinct. We obtain ten sets of numbers in descending order without overlapping. It will be noticed that the symbol $>$ invariably occurs in correspondence with one of the numbers which, when added together, become the greater index of the permutation. We have now to form the ten generating functions which enumerate the ten sets of numbers.

To enumerate set No. 6, put

$$\begin{aligned} b_2 &= a_3+A+1, \\ a_2 &= a_3+A+B+1, \\ b_1 &= a_3+A+B+C+1+1, \\ a_1 &= a_3+A+B+C+D+1+1, \end{aligned}$$

where A, B, C, D are arbitrary positive (including zero) integers; then

$$\Sigma x^{a_1+a_2+a_3+b_1+b_2} = \Sigma x^{5a_3+4A+3B+2C+D+2+4},$$

where the two numbers 2, 4, which present themselves in the exponent of x, are necessarily the two numbers which are added together to form the greater index of the associated permutation $\alpha\beta\alpha\beta\alpha$. This generating function is therefore

$$\frac{x^6}{(1)\,(2)\,(3)\,(4)\,(5)};$$

and it is readily seen that the sum of the ten generating functions must be

$$\frac{\Sigma x^p}{(1)\,(2)\,(3)\,(4)\,(5)},$$

the sum being in respect of the greater indices of the ten permutations. Hence

$$\frac{\Sigma x^p}{(1)\,(2)\,(3)\,(4)\,(5)} = \frac{1}{(1)\,(2)\,(3)\cdot(1)\,(2)},$$

or

$$\Sigma x^p = \frac{(1)\,(2)\,(3)\,(4)\,(5)}{(1)\,(2)\,(3)\cdot(1)\,(2)}.$$

The above reasoning is of general application. When the system of nodes has rows containing i, j, k, ... nodes and we consider the number of ways of placing numbers in descending orders in each row so that the sum of all the numbers may be w, we have to consider

$$\frac{(i+j+k+\ldots.)\,!}{i\,!\,j\,!\,k\,!\ldots.}$$

conditioned descending orders corresponding to the permutations of $\alpha^i\beta^j\gamma^k\ldots.$; whenever the contact is major we employ the symbol $>$, and $\geq$ in other cases, as conditions of the descending order, and we find in each case that x^p is the numerator of the generating function, where p is the greater index of the permutation. The reader will note that when the p_1-th interval between the letters $a_1, a_2, a_3, \ldots.$; $b_1, b_2, \ldots.$; $c_1, c_2, c_3, \ldots.$; ...., in appropriate order, is occupied by the symbol $>$, one outstanding number, in the exponent of x which first arises, is p_1; and that the whole outstanding number must be p.

Hence in general

$$\Sigma x^p = \frac{(1)(2)\ldots.(\mathbf{i}+\mathbf{j}+\mathbf{k}+\ldots.)}{(1)(2)\ldots.(\mathbf{i})\cdot(1)(2)\ldots.(\mathbf{j})\cdot(1)(2)\ldots.(\mathbf{k})\cdot\ldots.}\,.$$

7. When x is put equal to unity the generating function naturally reduces to the number which enumerates the permutations of the assemblage.

8. *Important Property of* Σx^p. From the symmetry of the expression just obtained we gather the information that the expression of Σx^p is unaltered by any substitution impressed upon the numbers $i, j, k, \ldots.$

Ex. gr., Σx^p is the same for $\alpha^3\beta^2$ as for $\alpha^2\beta^3$.

9. *Proof that* $\Sigma x^r = \Sigma x^p$. The last remark enables us to establish that in general $\Sigma x^r = \Sigma x^p$. Consider a permutation of any assemblage, viz.

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma,$$

and transfrom it by the substitution

$$\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix}$$

(equivalent to changing the assemblage $\alpha^4\beta^2\gamma^3$ into $\alpha^3\beta^2\gamma^4$). The permutation becomes

$$\beta\gamma\gamma\gamma\alpha\alpha\beta\gamma\alpha,$$

and its lesser index is necessarily equal to the greater index of the original permutation. Hence Σx^r for the transformed is equal to Σx^p for the original assemblage; and since, as we have seen above, Σx^p remains unchanged by the transformation, it follows at once that for every assemblage

$$\Sigma x^r = \Sigma x^p;$$

for we have merely to make the substitution

$$\begin{pmatrix}\alpha_1\alpha_2 & \alpha_3 & \ldots.\alpha_s\\ \alpha_s\alpha_{s-1} & \alpha_{s-2} & \ldots.\alpha_1\end{pmatrix},$$

and employ the same reasoning.

10. *The Expression of* Σx^{p+q}. Of course p, q are the greater and equal indices of the same permutation. We have

$$p+q+r = 1+2+\ldots.+(\Sigma i-1) = \binom{\Sigma i}{2},$$

p, q, r referring to the same permutation, and $\binom{\Sigma i}{2}$ denoting $\frac{1}{2}\Sigma i(\Sigma i-1)$. Hence

$$\Sigma x^{p+q} = x^{p+q+r}\Sigma x^{-r}$$

$$= x^{\binom{\Sigma i}{2}} \frac{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots.\left(1-\frac{1}{x^{i_1+i_2+\ldots.}}\right)}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots.\left(1-\frac{1}{x^{i_1}}\right)\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots.\left(1-\frac{1}{x^{i_2}}\right)\ldots.}$$

$$= x^{\binom{\Sigma i}{2}+\binom{i_1+1}{2}+\binom{i_2+1}{2}+\ldots.-\binom{\Sigma i+1}{2}} \frac{(1)(2)\ldots.(\mathbf{i}_1+\mathbf{i}_2+\ldots.)}{(1)(2)\ldots.(\mathbf{i}_1)\cdot(1)(2)\ldots.(\mathbf{i}_2)\cdot\ldots.};$$

and thence, on reduction,

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\Sigma x^{p},$$

a noteworthy result, the truth of which can also be seen *à priori.*

We have merely to recur to the method by which the expression of Σx^p was investigated. In the descending orders of numbers associated with the permutations we must clearly have the symbol $>$ *both* when there are major and when there are equal contacts. This will happen if we impress a condition upon the numbers in each row. The condition is that in any row of numbers, placed at the nodes of a row of nodes, there must be no two numbers equal. This is secured by adding to every row of numbers

$$a_1 a_2 a_3 \ldots . a_i$$

the numbers $i-1$, $i-2$, $i-3$, ..., 1, 0 respectively to each number in succession from left to right, so that the row becomes

$$a_1+i-1,\ a_2+i-2,\ a_3+i-3,\ \ldots.,\ a_{i-1}+1,\ a_i.$$

The result of this is to multiply Σx^p by $x^{1+2+\cdots+i-1}$ or by $x^{\binom{i}{2}}$ in respect of each row. Thence at once

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\Sigma x^{p}.$$

11. *Difference Equation for* Σx^p. Taking the assemblage to be $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$ write

$$\Sigma x^p = u(i_1,\ i_2,\ \ldots.,\ i_s)\ ;$$

and if Σx^p is restricted to those permutations which terminate with a letter α_m, write

$$\underset{m}{\Sigma x^p} = \underset{m}{u}(i_1,\ i_2,\ \ldots.,\ i_s), \quad \text{and} \quad \Sigma i = S;$$

so that

$$\Sigma x^p = \overset{s}{\underset{1}{\Sigma}}\,\underset{m}{u}(i_1,\ i_2,\ \ldots.,\ i_s).$$

Obviously

$$\underset{s}{u}(i_1 i_2 \ldots . i_s) = u(i_1,\ i_2,\ \ldots.,\ i_{s-1},\ i_s-1).$$

The function $\underset{s-1}{u}(i_1 i_2 \ldots . i_s)$ refers to permutations that may end with $\alpha_s\alpha_{s-1}$. These give rise to the function $x^{S-1}\underset{s}{u}(i_1,\ i_2,\ \ldots.,\ i_{s-1}-1,\ i_s)$. The remaining permutations yield the function

$$u(i_1,\ i_2,\ \ldots.,\ i_{s-1}-1,\ i_s) - \underset{s}{u}(i_1,\ i_2,\ \ldots.,\ i_{s-1}-1,\ i_s).$$

Hence

$$\underset{s-1}{u}(i_1,\ i_2,\ \ldots.,\ i_s)$$
$$= u(i_1,\ i_2,\ \ldots.,\ i_{s-1}-1,\ i_s) - (1-x^{S-1})\,u(i_1,\ i_2,\ \ldots.,\ i_{s-1}-1,\ i_s-1).$$

Similarly it will be found that

$$\begin{aligned}\underset{s-2}{u}(i_1, i_2, \dots, i_s) = {} & u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(1-x^{S-1})u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & -(1-x^{S-1})u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & +(1-x^{S-1})(1-x^{S-2})u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}-1, i_s-1),\end{aligned}$$

and so forth. Thence

$$\begin{aligned}u(i_1, i_2, \dots, i_s) = {} & u(i_1, i_2, \dots, i_{s-1}, i_s-1) + \{u(i_1, i_2, \dots, i_{s-1}-1, i_s) \\ & -(\mathbf{S}-1)u(i_1, i_2, \dots, i_{s-1}-1, i_s-1)\} \\ & +\{u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(\mathbf{S}-1)u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & -(\mathbf{S}-1)u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & +(\mathbf{S}-1)(\mathbf{S}-2)u(i_1, i_2, \dots, i_{s-2}-1, i_{s-1}-1, i_s-1)\} + \dots,\end{aligned}$$

and we obtain without difficulty the general equation of differences

$$\begin{aligned}u(i_1, i_2, \dots, i_s) = {} & \sum_{l} u(i_1, \dots, i_l-1, \dots, i_s) \\ & -(\mathbf{S}-1)\sum_{l,m} u(i_1, \dots, i_l-1, \dots, i_m-1, \dots, i_s) \\ & +(\mathbf{S}-1)(\mathbf{S}-2)\sum_{l,m,n} u(i_1, \dots, i_l-1, \dots, i_m-1, \dots, i_n-1, \dots, i_s) \\ & -\dots\dots\end{aligned}$$

This relation will be of service in the sequel.

12. *The Particular Case of the Assemblage $\alpha^i\beta^j$.* Merely considering the main result for the permutations of $\alpha^i\beta^j$, viz.

$$\Sigma x^p = \frac{\mathbf{(1)(2)\dots(i+j)}}{\mathbf{(1)(2)\dots(i)\cdot(1)(2)\dots(j)}},$$

we notice that Σx^p is equal to the generating function which enumerates the partitions of all numbers into parts limited in magnitude by i and in number by j. This indicates that the number of permutations of $\alpha^i\beta^j$ which have p for greater index is equal to the number of partitions of p, the parts being restricted as above stated by the numbers i, j.

If the permutation is of class m (see Art. 2) the greater index is obtained as the sum of m numbers. If we restrict ourselves to such permutations, I showed, in the postscript of the paper above referred to, that

$$\overset{m}{\Sigma} x^p = x^{m^2}\frac{\mathbf{(i-m+1)(i-m+2)\dots(i)}}{\mathbf{(1)(2)\dots(m)}}\cdot\frac{\mathbf{(j-m+1)(j-m+2)\dots(j)}}{\mathbf{(1)(2)\dots(m)}},$$

a result which yields the well-known expansion

$$\frac{\mathbf{(1)(2)\dots(i+j)}}{\mathbf{(1)(2)\dots(i)\cdot(1)(2)\dots(j)}} = 1 + x\frac{\mathbf{(i)(j)}}{\mathbf{(1)\cdot(1)}} + x^4\frac{\mathbf{(i-1)(i)\cdot(j-1)(j)}}{\mathbf{(1)(2)\cdot(1)(2)}} + \dots$$

This fact is referred to here in order that I may make the observation that the expansion is identical with that which Sylvester obtained from Durfee's dissection of the graph of a partition into a square of nodes with lateral and subjacent appendages. (Compare *Collected Mathematical Papers*, Vol. IV, pp. 1 *et seq.*).

13. For the assemblage $\alpha^i\beta^j$ we can find the equal index function Σx^q. It has first to be shown that the number $p-r$ is equal to $-i$ or $+j$ according as the permutation terminates with β or α.

Assume the law to obtain for $\alpha^i\beta^j$ and denote by $(\alpha^i\beta^j)_\alpha$, $(\alpha^i\beta^j)_\beta$ any permutations terminating with α, β respectively.

$$\begin{array}{lll}
\text{Then for} & (\alpha^i\beta^j)_\alpha\alpha, & p-r=+j; \\
\text{for} & (\alpha^i\beta^j)_\alpha\beta, & p-r=+j-(i+j)=-i; \\
\text{for} & (\alpha^i\beta^j)_\beta\alpha, & p-r=-i+i+j=+j; \\
\text{for} & (\alpha^i\beta^j)_\beta\beta, & p-r=+j; \\
\text{for} & (\alpha^i\beta^j)_\alpha\alpha\beta, & p-r=+j-(i+j+1)=-i-1; \\
\text{for} & (\alpha^i\beta^j)_\alpha\beta\alpha, & p-r=+j-(i+j)+(i+j+1)=+j+1; \\
\text{for} & (\alpha^i\beta^j)_\beta\alpha\beta, & p-r=-i+(i+j)-(i+j+1)=-i-1; \\
\text{for} & (\alpha^i\beta^j)_\beta\beta\alpha, & p-r=-i+(i+j+1)=+j+1.
\end{array}$$

These results establish the theorem.

Hence, for the permutations ending with β,

$$\Sigma x^{p-r}=\binom{i+j-1}{i}x^{-i};$$

and, for those ending with α,

$$\Sigma x^{p-r}=\binom{i+j-1}{j}x^{j}.$$

Since, for all permutations,

$$\Sigma x^{p+q+r}=\binom{i+j}{i}x^{\binom{i+j}{2}},$$

it follows that

$$\Sigma x^{2p+q}=\binom{i+j-1}{i}x^{\binom{i+j}{2}-i}+\binom{i+j-1}{j}x^{\binom{i+j}{2}+j}.$$

Writing Σx^p, for the whole of the permutation, equal to $F_{i,j}(x)$, we see that, for those ending with β, α,

$$\Sigma x^p=F_{i,j-1}(x) \quad \text{and} \quad x^jF_{i-1,j}(x)$$

respectively. (Compare Art. 11.) Hence

$$\Sigma x^q=\Sigma x^{2p+q-2p}=x^{\binom{i+j}{2}-i}F_{i,j-1}\left(\frac{1}{x^2}\right)+x^{\binom{i+j}{2}+j-2j}F_{i-1,j}\left(\frac{1}{x^2}\right);$$

easily reducing to the formula

$$\Sigma x^q = x^{\binom{i-j+1}{2}} F_{i,j-1}(x^2) + x^{\binom{j-i+1}{2}} F_{i-1,j}(x^2).$$

It will be noted that the index q is *always* even when $i-j \equiv 0 \bmod 4$ and *always* uneven when $i-j \equiv 2 \bmod 4$.

14. We may obtain the expression of Σx^{p+r}; for this is

$$\Sigma x^{p-r+2r};$$

and, since Σx^r is $x^i F_{i,j-1}(x)$ or $F_{i-1,j}(x)$ according as the permutation ends with β or α,

$$\begin{aligned}\Sigma x^{p+r} &= x^{-i} \cdot x^{2i} F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2),\\ &= x^i F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2).\end{aligned}$$

The reader will have no difficulty in obtaining the expression of $\Sigma x^{Ap+Bq+Cr}$ for this assemblage.

15. Resuming the consideration of the assemblage $\alpha^i\beta^j\gamma^k \ldots.$, it is to be remarked that $p-r$ has no longer any simple relation to the permutations when the number of Greek letters exceeds 2. It is not possible to obtain expressions for Σx^q and Σx^{p+r} in the foregoing manner. Similar theorems, however, will be reached when major and minor indices come under discussion later in the paper. Before proceeding to those matters I give some interesting properties of the indices that have been already under discussion.

16. *The Parity of the Greater Index.* The permutations of $\alpha^i\beta^j\gamma^k \ldots.$ are such that the number with an even index is equal to the number with an uneven index whenever more than one of the numbers $i, j, k, \ldots.$ are uneven. For consider the function

$$\frac{(1)(2)\ldots.(i+j+k+\ldots.)}{(1)(2)\ldots.(i)\cdot(1)(2)\ldots.(j)\cdot(1)(2)\ldots.(k)\cdot\ldots.};$$

if we put therein $x=-1$ and the function happens to vanish it must be because the permutations with even index are equal in number to those with uneven index. The factor (m) becomes 2 for $x=-1$ whenever m is uneven. Eliminating these factors we are left with a function which may be an indeterminate form and may be evaluated when $x=-1$. This happens when numerator and denominator contain the same number of factors. If this be not the case, the numerator must contain *more* factors than the denominator and the function must therefore vanish for $x=-1$. The numerator cannot contain less factors than the denominator, for in that case the function would become infinite for $x=-1$ and this result, from the nature of the function, is absurd. We have then to find out the circumstances under which the numerator has more factors than the denominator after the elimination of factors referred to above. If $E\sigma$ denotes the greatest integer in σ, the difference in number between the numerator and and denominator factors is

$$E\frac{1}{2}(i+j+k+\dots)-E\frac{i}{2}-E\frac{j}{2}-E\frac{k}{2}-\dots;$$

and this is greater than zero whenever more than one of the numbers $i, j, k, \dots$ are uneven. This establishes the theorem.

17. There is an analogous result in the case of every *prime* modulus μ. For, putting x equal to ωx, where ω is an imaginary μ-th root of unity, we set apart all factors which involve ω^t, where t is not equal to μ or $\equiv 0 \bmod \mu$. We are left with a number of factors free from ω, and the function involving them is seen to vanish when $x=\omega$, whenever

$$\Sigma\frac{1}{\mu}(i+j+k+\dots)-E\frac{i}{\mu}-E\frac{j}{\mu}-E\frac{k}{\mu}-\dots>0.$$

This happens if $i\equiv i' \bmod \mu$, $j\equiv j' \bmod \mu$, $k\equiv k' \bmod \mu$, etc., whenever

$$i'+j'+k'+\dots>\mu-1.$$

Under these circumstances the number of permutations, which have indices $\equiv \sigma \bmod \mu$, does not vary by variation of σ.

Ex. gr., take the assemblage $\alpha^2\beta\gamma$:

	$p=$		$p=$
$\alpha\alpha\beta\gamma$	0	$\beta\alpha\alpha\gamma$	1
$\alpha\alpha\gamma\beta$	3	$\gamma\alpha\alpha\beta$	1
$\alpha\beta\alpha\gamma$	2	$\beta\alpha\gamma\alpha$	4
$\alpha\gamma\alpha\beta$	2	$\gamma\alpha\beta\alpha$	4
$\alpha\beta\gamma\alpha$	3	$\beta\gamma\alpha\alpha$	2
$\alpha\gamma\beta\alpha$	5	$\gamma\beta\alpha\alpha$	3

The reader will verify that the even and uneven indices are equal in number; also, since 2, 1, 1 are $\equiv$ 2, 1, 1 mod 3 respectively and $2+1+1>3$, that the indices $\equiv$ 0, 1, 2 mod 3 are equal in number.

18. *Average Values of the Indices.* The expression obtained for Σx^p enables us to find the average values of the different indices of a permutation.

Write $\Sigma x^p=\Sigma C_p x^p$, so that C_p permutations have the greater index equal to p. The average value of p is the quotient of $\Sigma C_p p$ by

$$\frac{(i+j+k+\dots)!}{i!\,j!\,k!\,\dots}.$$

To find $\Sigma C_p p$ we may differentiate Σx^p with regard to x and then put x

equal to unity. This is not the only, or the most obvious, way of finding $\Sigma C_p p$, but it is the best way for present purposes, as will appear later. To evaluate

$$\left\{\partial_x \frac{(1)(2)\ldots\ldots(i+j)}{(1)(2)\ldots\ldots(i)\cdot(1)(2)\ldots\ldots(j)}\right\}_{x=1},$$

we require the value of

$$\left\{\partial_x \frac{(i+\sigma)}{(\sigma)}\right\}_{x=1}.$$

This is readily found to be

$$\frac{i(i+\sigma)}{2\sigma},$$

so that

$$\left\{\partial_x \frac{(i+1)\ldots\ldots(i+j)}{(1)\ldots\ldots(j)}\right\}_{x=1} = \frac{(i+j)\,!}{i\,!\,j\,!}\sum_{\sigma=1}^{\sigma=j}\frac{\dfrac{i(i+\sigma)}{2\sigma}}{\dfrac{i+\sigma}{\sigma}} = \frac{(i+j)\,!}{i\,!\,j\,!}\cdot\frac{1}{2}ij;$$

establishing that the average value of the greater (or lesser) index of the permutation of $\alpha^i\beta^j$ is

$$\frac{1}{2}ij.$$

Thence

$$\left\{\partial_x \frac{(1)(2)\ldots(i+j+k)}{(1)(2)\ldots(i)\cdot(1)(2)\ldots(j)\cdot(1)(2)\ldots(k)\cdot\ldots}\right\}_{x=1}$$

$$=\left\{\frac{(1)(2)\ldots(i+j+k)}{(1)(2)\ldots(i)\cdot(1)(2)\ldots(j+k)}\right\}_{x=1}\left\{\partial_x \frac{(1)(2)\ldots(j+k)}{(1)(2)\ldots(j)\cdot(1)(2)\ldots(k)}\right\}_{x=1}$$

$$+\left\{\frac{(1)(2)\ldots(j+k)}{(1)(2)\ldots(j)\cdot(1)(2)\ldots(k)}\right\}_{x=1}\left\{\partial_x \frac{(1)(2)\ldots(i+j+k)}{(1)(2)\ldots(i)\cdot(1)(2)\ldots(j+k)}\right\}_{x=1}$$

$$=\frac{(i+j+k)\,!}{i\,!\,j\,!\,k\,!}\left\{\frac{1}{2}jk+\frac{1}{2}i(j+k)\right\};$$

establishing that the average value of the greater (or lesser) index of the permutations of $\alpha^i\beta^j\gamma^k$ is

$$\frac{1}{2}(ij+ik+jk).$$

From the above it is quite clear that, for the permutations of $\alpha^i\beta^j\gamma^k\ldots\ldots$, the average value of the greater (or lesser) index is

$\frac{1}{2}$ (sum of the products two and two together of the numbers $i, j, k, \ldots\ldots$).

19. *Average Value of the Equal Index.* From the general formula

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\,\Sigma x^p,$$

it is seen that the average value of the equal index is

$$\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\dots\dots;$$

or from the circumstance that the average value of $p+r$ is

$$ij+ik+jk+\dots\dots,$$

and that the sum $p+q+r$ is invariably

$$\binom{i+j+k+\dots\dots}{2},$$

the average value of q is

$$\binom{i+j+k+\dots\dots}{2}-(ij+ik+jk+\dots\dots),$$

which is

$$\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\dots\dots.$$

20. *Average Value of the Square of the Greater Index.* The value in question is derived from the average value of $p(p-1)$, which is clearly the quotient of

$$\left\{\partial_x^2\frac{(\mathbf{1})(\mathbf{2})\dots\dots(\mathbf{i}+\mathbf{j}+\mathbf{k}+\dots\dots)}{(\mathbf{1})(\mathbf{2})\dots\dots(\mathbf{i})\cdot(\mathbf{1})(\mathbf{2})\dots\dots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\dots\dots(\mathbf{k})\cdot\dots\dots}\right\}_{x=1}$$

by

$$\frac{(i+j+k+\dots\dots)!}{i!\,j!\,k!\dots\dots}.$$

To evaluate this we require the value of

$$\left\{\partial_x^2\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}$$

when $\lambda>\mu$. This I find to be

$$\frac{1}{6}\frac{\lambda}{\mu}(\lambda-\mu)(2\lambda-\mu-3).$$

There is theoretically no difficulty in determining the average value of any given positive integral power of the greater index. It depends upon finding the value of

$$\left\{\partial_x^s\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}$$

when $\lambda>\mu$.

The simplest cases, corresponding to $s=1$ and $s=2$, may be written

$$\left\{\partial_x\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

$$\left\{\partial_x^2\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}-\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

and the general result, at which I have arrived, may be stated in the following manner: Let

$$\mu(\mu-1)(\mu-2)\ldots\ldots(\mu-s+1)=\mu^s+A_{s-1}\mu^{s-1}+A_{s-2}\mu^{s-2}+\ldots\ldots+A_1\mu,$$

and

$$1^\nu+2^\nu+\ldots\ldots+n^\nu={}_nS_\nu,$$

then

$$\left\{\partial_x^s\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\left\{\mu^s{}_nS_s+A_{s-1}\mu^{s-1}{}_nS_{s-1}+A_{s-2}\mu^{s-2}{}_nS_{s-2}+\ldots\ldots+A_1\mu\,{}_nS_1\right\}_{n=\frac{\lambda}{\mu}-1}.$$

I do not interrupt the present investigation by giving a proof of this elegant theorem. I write at length the two cases which follow those given above.

$$\left\{\partial_x^3\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^3\left\{\frac{1}{4}\frac{\lambda^2}{\mu^2}\left(\frac{\lambda}{\mu}-1\right)^2\right\}-3\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}+2\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

$$\left\{\partial_x^4\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^4\left\{\frac{1}{30}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\left(3\frac{\lambda^2}{\mu^2}-3\frac{\lambda}{\mu}-1\right)\right\}-6\mu^3\left\{\frac{1}{4}\frac{\lambda^2}{\mu^2}\left(\frac{\lambda}{\mu}-1\right)^2\right\}$$
$$+11\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}-6\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\}.$$

These formulæ may be often simplified. Thus I find that

$$\left\{\partial_x^3\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\frac{1}{4}\frac{\lambda}{\mu}(\lambda-\mu)(\lambda-\mu-2)(\lambda-2).$$

The result depends upon the sums of the powers of the natural numbers and therefore ultimately upon the numbers of Bernouilli. I notice that

$${}_{-n}S_\nu=(-)^{\nu+1}{}_{n-1}S_\nu,$$

a useful relation. Writing

$$\left\{\partial_x^s\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=K_s,\quad\text{and}\quad({}_nS_\nu)_{n=\frac{\lambda}{\mu}-1}=S'_\nu,$$

we have

$$\begin{array}{ll}
K_1=\mu S'_1, & \mu S'_1=K_1, \\
K_2=\mu^2S'_2-\mu S'_1, & \mu^2S'_2=K_1+K_2, \\
K_3=\mu^3S'_3-3\mu^2S'_2+2\mu S'_1, & \mu^3S'_3=K_1+3K_2+K_3, \\
K_4=\mu^4S'_4-6\mu^3S'_3+11\mu^2S'_2-6\mu S'_1, & \mu^4S'_4=K_1+7K_2+6K_3+K_4, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{array}$$

where the coefficients in the expression of K_s are derived from the product $x(x-1)\ldots\ldots(x-s+1)$, and those in the expression of $\mu^sS'_s$ from the expression of x^s as a linear function of x, $x(x-1)$, $x(x-1)(x-2)$, ...

To apply these results I write

$$(X-i)(X-j)(X-k)\ldots\ldots=X^s-\{1\}X^{s-1}+\{11\}X^{s-2}-\ldots\ldots$$

I find that the average value of $p(p-1)$ is

$$\frac{1}{12}[3\{22\}+6\{211\}+18\{1111\}+\{21\}+2\{111\}-5\{11\}],$$

where, in the notation of symmetric functions, $\{22\}$ denotes $\Sigma i^2 j^2$, etc. Since we have established that the average value of p is $\frac{1}{2}\{11\}$, we see that the average value of p^2 is

$$\frac{1}{12}[3\{22\}+6\{211\}+18\{1111\}+\{21\}+2\{111\}+\{11\}],$$

which may be written

$$\frac{1}{4}\{11\}^2+\frac{1}{12}[\{21\}+2\{111\}+\{11\}],$$

wherein it will be noticed that the first term is the square of the average value of p.

As a verification observe that, for the permutations of $\alpha\beta\gamma\delta$,

$$\Sigma x^p=1+3x+5x^2+6x^3+5x^4+3x^5+x^6;$$

so that the sum of the squares of the indices is

$$3\cdot 1^2+5\cdot 2^2+6\cdot 3^2+5\cdot 4^2+3\cdot 5^2+1\cdot 6^2=268,$$

and the average value of the square $=\frac{268}{24}=\frac{67}{6}$.

The formula gives $\frac{1}{12}(3\cdot 6+6\cdot 12+18\cdot 1+12+2\cdot 4+6)=\frac{76}{6}$.

For the case $\alpha^i\beta^j$ the average value of the square of p is

$$\frac{1}{12}ij(3ij+i+j+1).$$

21. *Average Value of the Cube of the Greater Index.* Similarly the reader will find with little difficulty that the average value of $p(p-1)(p-2)$ for the assemblage $\alpha^i\beta^j$ is

$$\frac{1}{8}ij(i^2j^2+i^2j+ij^2-5ij-2i-2j+6);$$

and; since $p^3=p+3p(p-1)+p(p-1)(p-2)$, we find the average value of p^3 by adding to the above

$$\frac{1}{2}ij+\frac{1}{4}ij(3ij+i+j-5);$$

thence the average value of p^3 is found to be

$$\frac{1}{8}i^2j^2(i+1)(j+1).$$

22. For the assemblage $\alpha^i\beta^j\gamma^k$ if we write

$$\frac{(1)(2)\ldots.(\mathrm{i+j+k})}{(1)(2)\ldots.(\mathrm{i})\cdot(1)(2)\ldots.(\mathrm{j})\cdot(1)(2)\ldots.(\mathrm{k})}$$

$$=\frac{(1)(2)\ldots.(\mathrm{i+j+k})}{(1)(2)\ldots.(\mathrm{i})\cdot(1)(2)\ldots.(\mathrm{j+k})}\cdot\frac{(1)(2)\ldots.(\mathrm{j+k})}{(1)(2)\ldots.(\mathrm{j})\cdot(1)(2)\ldots.(\mathrm{k})}$$

$$=F_{i,\,j+k}\cdot F_{j,\,k},$$

since

$$\partial_x^3 F_{i,j,k}=F_{i,j+k}\,\partial_x^3 F_{j,k}+3\,\partial_x F_{i,j+k}\,\partial_x^2 F_{j,k}+3\,\partial_x^2 F_{i,j+k}\,\partial_x F_{j,k}+\partial_x^3 F_{i,j+k}\cdot F_{j,k},$$

the average value of $p(p-1)(p-2)$ is seen to be

$$\frac{1}{8}jk(j^2k^2+j^2k+jk^2-5jk-2j-2k+6)+\frac{3}{2}i(j+k)\cdot\frac{1}{12}jk(3jk+j+k-5)$$

$$+3\cdot\frac{1}{12}i(j+k)\{3i(j+k)+i+j+k-5\}\cdot\frac{1}{2}jk$$

$$+\frac{1}{8}i(j+k)\{i^2(j+k)^2+i^2(j+k)+i(j+k)^2-5i(j+k)-2i-2(j+k)+6\},$$

which is

$$\frac{1}{8}\{\Sigma i^3j^3+3\Sigma i^3j^2k+6\Sigma i^2j^2k^2+\Sigma i^3j^2+2\Sigma i^3jk+4\Sigma i^2j^2k$$

$$-5\Sigma i^2j^2-10\Sigma i^2jk-2\Sigma i^2j-4ijk+6\Sigma ij\}.$$

Recalling that, for the same assemblage,

Average value of p is $\frac{1}{2}\Sigma ij$,

and of $p(p-1)$ is $\frac{1}{12}(3\Sigma i^2j^2+6\Sigma i^2jk+\Sigma i^2j+2ijk-5\Sigma ij)$,

we find that the average value of p^3 is

$$\frac{1}{8}\{\Sigma i^3j^3+3\Sigma i^3j^2k+6i^2j^2k^2+\Sigma i^3j^2+2\Sigma i^3jk+4\Sigma i^2j^2k+2\Sigma i^2jk+\Sigma i^2j^2\}.$$

23. With small labor I find that for the general assemblage the average value of p^3 is

$$\frac{1}{8}[6\{3111\}+15\{2211\}+\{33\}+3\{321\}+6\{222\}+9\{2111\}$$

$$+\{32\}+2\{311\}+4\{221\}+\{22\}+2\{211\}+6\{1111\}].$$

24. *Average Values from Another Standpoint.* For the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$ let us inquire into the effect which the particular major contact $\alpha_\mu\alpha_\lambda$ $(\mu>\lambda)$ has upon the sum of the greater indices Σp. When the contact has σ letters to the left of it, it adds $\sigma+1$ to the index of the permutation. This happens in the case of

$$\frac{(S-2)\,!}{i_1\,!\,i_2\,!\ldots.i_s\,!}\,i_\lambda i_\mu \text{ permutations, where } S=\Sigma\,i,$$

so that, altogether, the effect of the contact $\alpha_\mu\alpha_\lambda$ is to add to Σp the number

$$\sum_{\sigma=0}^{\sigma=S-2}\frac{(S-2)\,!}{i_1\,!\,i_2\,!\ldots.i_s\,!}\,i_\lambda i_\mu\,(\sigma+1),$$

or

$$\frac{S\,!}{i_1\,!\,i_2\,!\ldots.i_s\,!}\,\frac{1}{2}\,i_\lambda i_\mu.$$

Hence the contact $\alpha_\mu\alpha_\lambda$, on the average, adds the number $\frac{1}{2}\,i_\lambda i_\mu$ to the index of a permutation, and thence we find that the average value of p is

$$\frac{1}{2}\,\Sigma\,i_\lambda i_\mu,$$

verifying the conclusion of Art. 18.

25. The number p is obtained as the sum of m numbers $p_1,\,p_2,\,\ldots.,\,p_m$,[*] where m is the class of the permutation *quâ* the greater index. If we had under consideration the sum

$$\Sigma\,(p_1^\nu+p_2^\nu+\ldots.+p_m^\nu),$$

we might inquire into the effect which the particular contact $\alpha_\mu\alpha_\lambda$ has upon such sum. The answer to this question is that the effect is to add to the sum the number

$$\sum_{\sigma=0}^{\sigma=S-2}\frac{(S-2)\,!}{i_1\,!\,i_2\,!\ldots.i_s\,!}\,i_\lambda i_\mu\,(\sigma+1)^\nu,$$

which is

$$\frac{S\,!}{i_1\,!\,i_2\,!\ldots.i_s\,!}\;\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,i_\lambda i_\mu.$$

Hence the contact $\alpha_\mu\alpha_\lambda$, on the average, adds to the sum

$$p_1^\nu+p_2^\nu+\ldots.+p_m^\nu,$$

with regard to a permutation, the number

$$\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,i_\lambda i_\mu.$$

From this result it follows that the average value of $p_1^\nu+p_2^\nu+\ldots.+p_m^\nu$ is

$$\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,\Sigma\,i_\lambda i_\mu.$$

26. The average value of $p_1^2+p_2^2+\ldots.+p_m^2$ is therefore

$$\frac{1}{6}\,(2\Sigma i_1-1)\,\Sigma\,i_1 i_2;$$

---

[*] These numbers it is convenient to term the "components" of the index p.

and of $p_1^3+p_2^3+p_3^3+\ldots\ldots+p_m^3$,

$$\frac{1}{4}\Sigma i_1(\Sigma i_1-1)\Sigma i_1 i_2\,;$$

and so forth.

To verify these results, take the assemblage $\alpha^2\beta\gamma$:

	p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$		p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$
$\alpha\alpha\beta\gamma$	0	0	0	0	$\beta\alpha\alpha\gamma$	1	0	1	1
$\alpha\alpha\gamma\beta$	3	0	9	27	$\gamma\alpha\alpha\beta$	1	0	1	1
$\alpha\beta\alpha\gamma$	2	0	4	8	$\beta\alpha\gamma\alpha$	1	3	10	28
$\alpha\gamma\alpha\beta$	2	0	4	8	$\gamma\alpha\beta\alpha$	1	3	10	28
$\alpha\beta\gamma\alpha$	3	0	9	27	$\beta\gamma\alpha\alpha$	2	0	4	8
$\alpha\gamma\beta\alpha$	2	3	13	35	$\gamma\beta\alpha\alpha$	1	2	5	9
			39	105				31	75

giving the average of $p_1^2+p_2^2$ and $p_1^3+p_2^3$ equal to $\frac{35}{6}$ and 15 respectively.

The formulæ give, since $i_1=2$, $i_2=1$, $i_3=1$, $\Sigma i_1=4$, $\Sigma i_1 i_2=5$,

$$\text{Average of } p_1^2+p_2^2=\frac{1}{6}\,(2\cdot 4-1)\cdot 5=\frac{35}{6},$$

$$\text{Average of } p_1^3+p_2^3=\frac{1}{4}\cdot 4\cdot 3\cdot 5=15,$$

a verification.

27. *Average Value of the Class of a Permutation quâ the Greater Index.* If in the foregoing results we put $\nu=0$, then

$$p_1^0+p_2^0+\ldots\ldots+p_m^0=m,$$

the class of the permutation. Hence the average value of the class is

$$\frac{S-1}{S(S-1)}\Sigma i_1 i_2 \text{ or } \frac{\Sigma i_1 i_2}{\Sigma i_1}.^*$$

In the above example

$$\frac{\Sigma i_1 i_2}{\Sigma i_1}=\frac{5}{4},$$

and we observe that one permutation is of class 0, seven of class 1 and four of class 2, so that the average class is

$$\frac{0+7+8}{12}=\frac{5}{4},$$

---

[*] Another corollary, paying attention to Art. 25, is that the average value of $p_g^\nu\,(p_g\neq 0)$ is

$$\frac{1}{\Sigma i-1}\left\{1^\nu+2^\nu+\ldots\ldots+(\Sigma i-1)^\nu\right\}.$$

a verification. In other words we may say that the average number of major contacts possessed by a permutation is

$$\frac{\Sigma i_1 i_2}{\Sigma i_1}.$$

28. An easy corollary is that a permutation possesses on the average

$$\frac{2\Sigma i_1 i_2}{\Sigma i_1}$$

contacts which are either major or minor.

Since, moreover, the whole number of contacts is $\Sigma i_1 - 1$, it follows that, on the average, a permutation possesses a number of equal contacts equal to

$$\Sigma i_1 - 1 - 2\frac{\Sigma i_1 i_2}{\Sigma i_1} \text{ or } 2\frac{\Sigma \binom{i_1}{2}}{\Sigma i_1},$$

an elegant result.

This last result may be obtained independently.

29. *Average Values Connected with the Equal Index.* The particular contact $\alpha_\lambda \alpha_\lambda$ has an effect upon the sum of the equal indices $\Sigma q = q_1 + q_2 + \ldots . + q_m$, where m is the class of the permutation *quâ* equal contacts. When such a contact has σ letters to the left of it, it adds $\sigma+1$ to the equal index of the permutation. This happens in the case of

$$\frac{(S-2)\,!}{i_1\,!\, i_2\,! \ldots . i_s\,!} \cdot 2\binom{i_\lambda}{2}$$

permutations; so that, altogether, the contact adds to Σq the number

$$\sum_{\sigma=0}^{\sigma=S-2} \frac{(S-2)\,!}{i_1\,!\, i_2\,! \ldots . i_s\,!} \cdot 2\binom{i_\lambda}{2}(\sigma+1),$$

or

$$\frac{S\,!}{i_1\,!\, i_2\,! \ldots . i_s\,!}\binom{i_\lambda}{2}.$$

Hence the contact $\alpha_\lambda \alpha_\lambda$, on the average, adds the number

$$\binom{i_\lambda}{2}$$

to the equal index of a permutation. Thence it appears that the average value of the equal index is

$$\Sigma\binom{i_1}{2}$$

(see Art. 19).

Moreover, the contact $\alpha_\lambda \alpha_\lambda$, on the average, adds to the sum

$$q_1^\nu + q_2^\nu + \ldots . + q_m^\nu$$

(where $q=q_1+q_2+\ldots\ldots+q_m$) the number

$$2\frac{1^\nu+2^\nu+\ldots\ldots+(S-1)^\nu}{S(S-1)}\binom{i_\lambda}{2};$$

and thence the average value of $q_1^\nu+q_2^\nu+\ldots\ldots+q_m^\nu$ is

$$2\cdot\frac{1^\nu+2^\nu+\ldots\ldots+(S-1)^\nu}{S(S-1)}\Sigma\binom{i_1}{2}.$$

The average values of $q_1^2+q_2^2+\ldots\ldots+q_m^2$ and $q_1^3+q_2^3+\ldots\ldots+q_m^3$ respectively are

$$\frac{1}{3}(2\Sigma i_1-1)\,\Sigma\binom{i_1}{2}$$

and

$$\frac{1}{2}\Sigma i(\Sigma i-1)\,\Sigma\binom{i_1}{2}.$$

Moreover, putting $\nu=0$, we find for the average class, *quâ* the equal index,

$$\frac{2}{\Sigma i_1}\Sigma\binom{i_1}{2},$$

agreeing with the result of Art. 28.

An easy corollary is that the average value of q_g^ν $(q_g\neq 0)$ is

$$\frac{1}{\Sigma i-1}\{1^\nu+2^\nu+\ldots\ldots+(\Sigma i-1)^\nu\}.$$

In fact q_g^ν and p_g^ν have the same average values in respect of those permutations for which they exist as positive non-zero numbers.

§ 2. *The Indices of the Second Genus.*

30. When we come to consider the permutations of $\alpha^i\beta^j\gamma^k$, we find that the value of $p-r$ does not follow any simple law, and moreover the only linear function of p, q and r that does so is the sum $p+q+r$. I now import a new idea into the subject by defining the major and minor indices of a permutation.

The Major Index of a Permutation. Consider two adjacent letters $\alpha_\mu\alpha_\lambda$ $(\mu>\lambda)$ of a permutation of the assemblage

$$\alpha_1^{i_1}\alpha_2^{i_2}\ldots\ldots\alpha_s^{i_s}.$$

If $\alpha_\mu\alpha_\lambda$ be the g-th major contact of the permutation, counting from the beginning, and α_μ be the p_g-th letter of the permutation, put

$$(\mu-\lambda)p_g=P_g.$$

Then $\sum\limits_{g=1}^{g=m}(\mu-\lambda)p_g$ or $\sum\limits_{g=1}^{g=m}P_g$ is the major index of the permutation and m is the class of the permutation *quâ* the major index. We put the major index equal to P, so that

$$P_1+P_2+\ldots\ldots+P_m=P.$$

31. *The Minor Index of a Permutation.* Consider two adjacent letters $\alpha_\lambda\alpha_\mu\ .(\mu>\lambda)$ of a permutation.

If $\alpha_\lambda\alpha_\mu$ be the g-th minor contact of the permutation, counting from the beginning, and α_λ be the r_g-th letter of the permutation, put

$$(\mu-\lambda)\,r_g=R_g\,.$$

Then $\sum\limits_{g=1}^{g=m}(\mu-\lambda)\,r_g$ or $\sum\limits_{g=1}^{g=m}R_g$ is the minor index of the permutation and m is the class of the permutation *quâ* the minor index. We put the minor index equal to R, so that

$$R_1+R_2+\ldots\ldots+R_m=R.$$

In this system of indices the equal index is zero since $(\mu-\lambda)$ vanishes for $\lambda=\mu$.

Ex. gr., for the permutation $\alpha_4\alpha_1\alpha_3\alpha_1\alpha_5$

$$\text{Major index}=3\cdot1+2\cdot3=9,$$
$$\text{Minor index}=2\cdot2+4\cdot4=20.$$

It will be at once noticed that for the particular assemblage $\alpha^i\beta^j$ the major and minor indices are the same as the greater and lesser indices respectively.

32. *Theorem in regard to P—R. We have now the important circumstance that, for the most general assemblage, the number P—R has only s values corresponding to the s different letters that may terminate the permutations.*

This will now be established. To bring something definite under the reader's eye I give the major and minor indices of the permutations of $\alpha\beta\gamma\delta$.

	P	R	$P-R$		P	R	$P-R$
$\alpha\beta\gamma\delta$	0	6	—6	$\alpha\beta\delta\gamma$	3	5	—2
$\alpha\gamma\beta\delta$	2	8	—6	$\alpha\delta\beta\gamma$	4	6	—2
$\beta\alpha\gamma\delta$	1	7	—6	$\beta\alpha\delta\gamma$	4	6	—2
$\beta\gamma\alpha\delta$	4	10	—6	$\beta\delta\alpha\gamma$	6	8	—2
$\gamma\alpha\beta\delta$	2	8	—6	$\delta\alpha\beta\gamma$	3	5	—2
$\gamma\beta\alpha\delta$	3	9	—6	$\delta\beta\alpha\gamma$	4	6	—2

	P	R	$P-R$		P	R	$P-R$
$\alpha\gamma\delta\beta$	6	4	2	$\beta\gamma\delta\alpha$	9	3	6
$\alpha\delta\gamma\beta$	5	3	2	$\beta\delta\gamma\alpha$	8	2	6
$\gamma\alpha\delta\beta$	8	6	2	$\gamma\beta\delta\alpha$	10	4	6
$\gamma\delta\alpha\beta$	6	4	2	$\gamma\delta\beta\alpha$	7	1	6
$\delta\alpha\gamma\beta$	6	4	2	$\delta\beta\gamma\alpha$	8	2	6
$\delta\gamma\alpha\beta$	5	3	2	$\delta\gamma\beta\alpha$	6	0	6

The symmetry of these numbers is striking. It gives a notion of the theorems to be established. Let

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s})$$

denote any permutation of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$, and let

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s})_l$$

denote any permutation of the same assemblage which terminates with α_l. Assume that, for every permutation

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s})_l,$$

$P-R$ has the value $-l\Sigma i+i_1+2i_2+3i_3+\ldots+si_s$. Now consider the value of $P-R$ when α_k is placed as terminal letter to the above permutation. We have three cases to consider.

(i) If $k<l$, P for $(\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s})_l\,\alpha_k$ becomes $P+(l-k)\Sigma i$ and R remains unaltered, so that $P-R$ becomes

$$-k(\Sigma i+1)+i_1+2i_2+\ldots+k(i_k+1)+\ldots+si_s,$$

and, since $\Sigma i+1$ and i_k+1 are the new values of Σi and i_k, we see that the value of $P-R$ is independent of l and thus obeys the law assumed.

(ii) If $k=l$, P and R are both unaltered and we may write the value of $P-R$

$$-l(\Sigma i+1)+i_1+2i_2+\ldots+l(i_l+1)+\ldots+si_s,$$

which is in accord with the assumed law.

(iii) If $k>l$, P is unchanged but R is increased by $(k-l)\Sigma i$, so that $P-R$ is diminished by $(k-l)\Sigma i$ and thus becomes

$$-k(\Sigma i+1)+i_1+2i_2+\ldots+k(i_k+1)+\ldots+si_s,$$

and again the assumed law is verified.

Therefore by induction from simple cases the general law that for the permutations of $(\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s})_l$ we have

$$P-R=-l\Sigma i+i_1+2i_2+\ldots+si_s$$

is established.

33. The law is, however, more extensive than this, for, instead of having letters $\alpha_1, \alpha_2, \ldots, \alpha_s$ with subscripts forming the series of natural numbers, the subscripts may be any numbers whatever. Thus, take the assemblage to be

$$\alpha_{m_1}^{i_{m_1}}\alpha_{m_2}^{i_{m_2}}\ldots\alpha_{m_s}^{i_{m_s}},$$

where $m_1, m_2, \ldots, m_s$ are *any* integers, of which one is l. Assume the law

$$P-R \text{ for } \left(\alpha_{m_1}^{i_{m_1}}\alpha_{m_2}^{i_{m_2}}\ldots\alpha_{m_s}^{i_{m_s}}\right)_l=-l\Sigma i+m_1i_{m_1}+m_2i_{m_2}+\ldots+li_l+\ldots+m_si_{m_s}.$$

Now adding α_k to the permutation as before we get for the new value of $P-R$

$$-k(\Sigma i+1)+m_1 i_{m_1}+m_2 i_{m_2}+\ldots\ldots+k(i_k+1)+\ldots\ldots+m_s i_{m_s},$$

whether k be $>$, $=$ or $<$ l and whether k is one of the numbers m_1, m_2, ..., m_s or not. This is independent of l and obeys the assumed law, so that the general law is established by induction. Setting forth the values of $P-R$ for the assemblages $\alpha_1^{i_1}\alpha_2^{i_2}$, $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}$, ... we obtain the schemes

$$\begin{array}{c|ll} \alpha_2 & -i_1 & \\ \alpha_1 & & +i_2, \end{array} \qquad \begin{array}{c|lll} \alpha_3 & -2i_1-i_2 & \\ \alpha_2 & -i_1 & +i_3 \\ \alpha_1 & & +i_2+2i_3, \end{array} \qquad \begin{array}{c|lll} \alpha_4 & -3i_1-2i_2-i_3 & & \\ \alpha_3 & -2i_1-i_2 & & +i_4 \\ \alpha_2 & -i_1 & +i_3 & +2i_4 \\ \alpha_1 & \quad\quad\ i_2 & +2i_3 & +3i_4, \end{array} \qquad \text{etc.,}$$

the letters α denoting the terminal letters of the permutations.

The values for the terminal letters α_m, α_{s-m+1} are symmetrically related. Each line of a scheme is derived from the preceding line by adding Σi.

34. *General Relation between Σx^P and Σx^R.* It is not true in general that $\Sigma x^P=\Sigma x^R$, although this equality happens to obtain for the particular case of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}$, as will be seen presently. The above investigation shows that if $\underset{l}{\Sigma} x^P$ refers to the permutations which terminate with α_l,

$$\underset{l}{\Sigma} x^P=x^{-l\Sigma i+i_1+2i_2+\ldots\ldots+si_s}\underset{l}{\Sigma} x^R.$$

This appears to be the most general relation obtainable, but, if the assemblage be unaltered by the substitution

$$\begin{pmatrix}\alpha_1\alpha_2 & \alpha_3 & \ldots\ldots\alpha_s \\ \alpha_s\alpha_{s-1} & \alpha_{s-2} & \ldots\ldots\alpha_1\end{pmatrix},$$

it is clear that the P and R of any permutation are the same as the R and P respectively of the permutation obtained by impressing the substitution. Thus in this case

$$\underset{l}{\Sigma} x^P=\underset{s-l+1}{\Sigma} x^R=x^{l\Sigma i-i_1-2i_2-\ldots\ldots-si_s}\underset{s-l+1}{\Sigma} x^P$$

and

$$\Sigma x^P=\Sigma x^R.$$

35. *Difference Equation for Σx^P.* Let

$$\Sigma x^P=v(i_1, i_2, \ldots\ldots, i_s),$$

and

$$\underset{m}{\Sigma} x^P=\underset{m}{v}(i_1, i_2, \ldots\ldots, i_s).$$

It is clear that $\underset{s}{v}(i_1, i_2, \ldots\ldots, i_s)=v(i_1, i_2, \ldots\ldots, i_{s-1}, i_s-1)$. The function $\underset{s-1}{v}(i_1, i_2, \ldots\ldots, i_s)$ refers to permutations that may end with $\alpha_s\alpha_{s-1}$; these give rise to the function

$$x^{\Sigma i-1}\underset{s}{v}(i_1, i_2, \ldots\ldots, i_{s-1}-1, i_s);$$

the remaining permutations give rise to the function

$$v(i_1, i_2, \ldots, i_{s-1}-1, i_s) - \underset{s}{v}(i_1, i_2, \ldots, i_{s-1}-1, i_s).$$

Hence

$$\underset{s-1}{v}(i_1 i_2 \ldots i_s) = v(i_1, i_2, \ldots, i_{s-1}-1, i_s) - (\Sigma\mathbf{i}-\mathbf{1})\, v(i_1, i_2, \ldots, i_{s-1}-1, i_s-1),$$

where $(\Sigma\mathbf{i}-\mathbf{1})$ means $1-x^{\Sigma i-1}$.

By similar reasoning we obtain

$$\begin{aligned} \underset{s-2}{v}(i_1 i_2 \ldots i_s) = {} & v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(\mathbf{2}\Sigma\mathbf{i}-\mathbf{2})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & -(\Sigma\mathbf{i}-\mathbf{1})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & +(\Sigma\mathbf{i}-\mathbf{1})(\Sigma\mathbf{i}-\mathbf{2})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s-1), \end{aligned}$$

and the general formula is easy to obtain, but we do not need it at present. An application will be made at once to a particular case.

36. *The Assemblage* $\alpha^i\beta^j\gamma^k$. Since $v(i,j,k) = \underset{3}{v}(i,j,k) + \underset{2}{v}(i,j,k) + \underset{1}{v}(i,j,k)$, we find

$$\begin{aligned} \Sigma x^P = v(i, j, k) = {} & v(i, j, k-1) + v(i, j-1, k) + v(i-1, j, k) \\ & -(\Sigma\mathbf{i}-\mathbf{1})\, v(i, j-1, k-1,) - (\Sigma\mathbf{i}-\mathbf{1})\, v(i-1, j-1, k) \\ & -(\mathbf{2}\Sigma\mathbf{i}-\mathbf{2})\, v(i-1, j, k-1) + (\Sigma\mathbf{i}-\mathbf{1})(\Sigma\mathbf{i}-\mathbf{2})\, v(i-1, j-1, k-1). \end{aligned}$$

Also interchanging i and k,

$$\begin{aligned} v(k, j, i) = {} & v(k, j, i-1,) + v(k, j-1, i) + v(k-1, j, i) \\ & -(\Sigma\mathbf{i}-\mathbf{1})\, v(k, j-1, i-1) - (\Sigma\mathbf{i}-\mathbf{1})\, v(k-1, j-1, i) \\ & -(\mathbf{2}\Sigma\mathbf{i}-\mathbf{2})\, v(k-1, j, i-1) + (\Sigma\mathbf{i}-\mathbf{1})(\Sigma\mathbf{i}-\mathbf{2})\, v(k-1, j-1, i-1). \end{aligned}$$

These results enable us to establish that

$$v(i, j, k) = v(k, j, i);$$

for if this law obtains in all cases when the sum of the three numbers i, j, k is less than w, it is seen that the right-hand sides of the expressions obtained for $v(i, j, k)$ and $v(k, j, i)$ are identical. If then we can show that the law obtains when $i+j+k$ is equal to any number w in all cases, the law must be universal.

We may note in the first place that

$$v(0, j, k) = \frac{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})} = F_{j,k}(x),$$

$$v(i, j, 0) = F_{i,j}(x), \quad v(i, 0, k) = F_{i,k}(x^2);$$

and that for these forms the law obtains. It follows that the law obtains when $w=2$, for then the forms are

$$v(2, 0, 0), \quad v(0, 2, 0), \quad v(0, 0, 2), \quad v(1, 1, 0), \quad v(1, 0, 1), \quad v(0, 1, 1).$$

Hence universally

$$v(i, j, k) = v(k, j, i).$$

The function is therefore symmetrical in i and k.

37. *Proof that for the Assemblage* $\alpha^i\beta^j\gamma^k$, $\Sigma x^P = \Sigma x^R$. Since the value of P for any permutation of the assemblage is equal to the value of R for the permutation obtained from the former by the substitution $(\alpha\gamma)$, it follows that Σx^P for the assemblage $\alpha^i\beta^j\gamma^k$ is equal to Σx^R for the assemblage $(\alpha^k\beta^j\gamma^i)$. Since, moreover, the substitution has been shown above not to alter the expression Σx^P, it follows that universally $\Sigma x^P = \Sigma x^R$.

38. *The Algebraic Expression of* Σx^P *for the Assemblage* $\alpha^i\beta^j\gamma^k$. Since it has been shown that the functions $u(0, j, k)$, $u(i, 0, k)$, $u(i, j, 0)$ are equal to $F_{j,k}(x)$, $F_{i,k}(x^2)$, $F_{i,j}(x)$ respectively, we are led to consider the function

$$\frac{(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{i+j+k})}{(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{i+k})}\cdot\frac{(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2i+2k})}{(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2i})\cdot(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2k})},$$

which possesses these properties and also becomes equal to the number of permutations of $\alpha^i\beta^j\gamma^k$ when $x=1$. It is, moreover, symmetrical in i and k, and is obviously correct in simple particular cases. *Ex. gr.*, it is $\dfrac{(\mathbf{3})(\mathbf{4})}{(\mathbf{1})(\mathbf{2})}$ when $i=j=k=1$.

In fact, it will be shown subsequently that

$$\Sigma x^P = F_{j,i+k}(x)\cdot F_{i,k}(x^2);$$

and it has been verified that this expression satisfies the difference equation for $v(i, j, k)$ obtained above.

Another verification of our results is obtained by showing algebraically that Σx^R has the same expression as Σx^P.

39. *Derivation of* Σx^R *from* Σx^P. Since $\Sigma x^R = \Sigma x^{R-P+P} = x^{R-P}\Sigma x^P$ for all sets of permutations for which $R-P$ has a constant value, we must have

$$\Sigma x^R = x^{2i+j}\underset{3}{v}(i, j, k) + x^{i-k}\underset{2}{v}(i, j, k) + x^{-j-2k}\underset{1}{v}(i, j, k);$$

and this will be found to be equal to the product of

$$\frac{(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{i+j+k-1})}{(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\ldots\ldots(\mathbf{i+k})}\cdot\frac{(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2i+2k-2})}{(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2i})\cdot(\mathbf{2})(\mathbf{4})\ldots\ldots(\mathbf{2k})}$$

and

$$x^{2i+j}(\mathbf{i+k})(\mathbf{2k}) + x^{i-k}(\mathbf{j})(\mathbf{2i+2k}) - x^{i-k}(\mathbf{j})(\mathbf{i+k})(\mathbf{2k})$$
$$+ x^{-j-2k}(\mathbf{i+k})(\mathbf{2i}) - x^{j-2k}(\mathbf{j})(\mathbf{i+k})(\mathbf{2i}) - x^{-2k}(\mathbf{i+k})(\mathbf{2i})(\mathbf{2k}).$$

The expression last written is found to be

$$(\mathbf{i+j+k})(\mathbf{2i+2k}),$$

establishing that

$$\Sigma x^R = \frac{(1)(2)\ldots\ldots(\mathrm{i}+\mathrm{j}+\mathrm{k})}{(1)(2)\ldots\ldots(\mathrm{j})\cdot(1)(2)\ldots\ldots(\mathrm{i}+\mathrm{k})}\cdot\frac{(2)(4)\ldots\ldots(2\mathrm{i}+2\mathrm{k})}{(2)(4)\ldots\ldots(2\mathrm{i})\cdot(2)(4)\ldots\ldots(2\mathrm{k})},$$

the expression already found for Σx^P.

40. *Evaluation of* Σx^{P+R}. We could also find the expression of $\Sigma x^{\lambda P+\mu R}$, where λ, μ are arbitrary positive or negative integers. To find Σx^{P+R} we consider

$$x^{R-P}\Sigma x^{2P}$$

in three fragments and if $\Sigma x^P = G_{i,\,j,\,k}(x)$, we find

$$\Sigma x^{P+R} = x^{2i+j}\underset{3}{G}_{i,\,j,\,k}(x^2) + x^{i-k}\underset{2}{G}_{i,\,j,\,k}(x^2) + x^{-j-2k}\underset{1}{G}_{i,\,j,\,k}(x^2),$$

where $\underset{t}{u}(i, j, k)$ is written $\underset{t}{G}_{i,\,j,\,k}(x)$.

I thence find

$$\Sigma x^{P+R} = \frac{(2)(4)\ldots\ldots(2\mathrm{i}+2\mathrm{j}+2\mathrm{k}-2)(+\mathrm{i}+\mathrm{j}+\mathrm{k})}{(2)(4)\ldots\ldots(2\mathrm{j})\cdot(2)(4)\ldots\ldots(2\mathrm{i}+2\mathrm{k}-2)}$$

$$\cdot\frac{(4)(8)\ldots\ldots(4\mathrm{i}+4\mathrm{k}-4)}{(4)(8)\ldots\ldots(4\mathrm{i})\cdot(4)(8)\ldots\ldots(4\mathrm{k})}\cdot(x^{2i}+x^{2k})(x^j+x^{i+k}).$$

Ex. gr., for $i=2$, $j=1$, $k=1$, I thus find

$$\Sigma x^{P+R} = x^3 + 3x^5 + 4x^7 + 3x^9 + x^{11},$$

which the reader can readily verify from the permutations of $\alpha^2\beta\gamma$.

41. *Average Values of the Major and Minor Indices.* Following the method of Art. 18, we have only to differentiate the major or minor index function of the assemblage $\alpha^i\beta^j\gamma^k$, viz.

$$F_{j,\,i+k}(x)\,F_{i,\,k}(x^2),$$

with regard to x, and then to put $x=1$, to obtain the value of ΣP or ΣR, and thence the mean value of P or R.

The mean value is in fact

$$\frac{\{\partial_x F_{j,\,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}} + \frac{\{\partial_x F_{i,\,k}(x^2)\}_{x=1}}{\binom{i+k}{i}},$$

which, making use of results obtained above, is found to be

$$\frac{1}{2}j(i+k)+ik$$

or

$$\frac{1}{2}(ij+2ik+jk).$$

42. The mean value of $P(P-1)$ or $R(R-1)$ is similarly found to be

$$\frac{\{\partial_x^2 F_{j,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}}+2\,\frac{\{\partial_x F_{j,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}}\cdot\frac{\{\partial_x F_{i,k}(x^2)\}_{x=1}}{\binom{i+k}{i}}+\frac{\{\partial_x^2 F_{i,k}(x^2)\}_{x=1}}{\binom{i+k}{i}};$$

and, making use of previous results, this is found to be

$$\frac{1}{12}\{12i^2k^2+4i^2k+4ik^2-8ik+j(12i^2k+12ik^2+i^2+k^2+2ik-5i-5k)$$
$$+j^2(3i^2+3k^2+6ik+i+k)\},$$

symmetrical in i, k, as should be the case.

Adding to the above the mean value of P, viz. $\frac{1}{2}(ij+2ik+jk)$, we find for the mean value of P^2 the expression

$$\frac{1}{12}\{4ik(3ik+i+k+1)+j(i+k)(12ik+i+k+1)+j^2(i+k)(3i+3k+1)\}.$$

There is of course no theoretical difficulty in obtaining the mean values of higher powers of P.

43. *Mean Values Discussed in the Second Manner.* Following the procedure taken with regard to the greater, equal and lesser indices, we can obtain some results which appertain to the major and minor indices of the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$.

I suppose that P is formed by addition in the manner

$$P=P_1+P_2+\ldots+P_m,$$

where m is the class of the permutation *quâ* the major index.

We inquire into the effect which the particular major contact $\alpha_{\mu+\nu}\alpha_\mu$ has upon the sum of the indices ΣP. When the contact has σ letters to its left and $\Sigma i-\sigma-2$ letters to its right, it adds $\nu(\sigma+1)$ to the index of the permutation. This happens in the case of

$$\frac{(\Sigma i-2)!\,i_\mu i_{\mu+\nu}}{i_1!\,i_2!\ldots i_s!}$$

permutations; so that, altogether, the contact adds to ΣP the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2}\frac{(\Sigma i-2)!\,i_\mu i_{\mu+\nu}}{i_1!\,i_2!\ldots i_s!}\nu(\sigma+1),$$

which is

$$\frac{1}{2}\nu\frac{(\Sigma i)!}{i_1!\,i_2!\ldots i_s!}i_\mu i_{\mu+\nu}.$$

Hence, on the average, it adds the number

$$\frac{1}{2}\nu\, i_\mu i_{\mu+\nu}$$

to the major index of a permutation.

Also the $s-\nu$ contacts $\alpha_{\nu+1}\alpha_1, \alpha_{\nu+2}\alpha_2, \ldots, \alpha_s\alpha_{s-\nu}$ add, on the average, the number

$$\frac{1}{2}\nu\,(i_1 i_{\nu+1}+i_2 i_{\nu+2}+\ldots+i_{s-\nu}i_s)$$

to the index.

Now making ν vary from 1 to $s-1$, we find that the average value of the major index is

$$\frac{1}{2}\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda)\,i_\lambda i_\mu,$$

where $\mu>\lambda$, a result which clearly embraces and generalizes that already found for the assemblage $\alpha^i\beta^j\gamma^k$ from other principles.

44. The result is essentially the average value of $P_1+P_2+\ldots+P_m$. To find the average value of $P_1^t+P_2^t+\ldots+P_m^t$, we have merely to note that, under circumstances such as the foregoing, the contact $\alpha_{\mu+\nu}\alpha_\nu$ adds the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2}\frac{(\Sigma i-2)!\,i_\mu i_{\mu+\nu}}{i_1!\,i_2!\ldots i_s!}\nu^t(\sigma+1)^t$$

to the sum ΣP_1^t. Hence we readily find that

$$\text{Average value of } \Sigma P_1^2=\frac{1}{6}(2\Sigma i-1)\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda)^2 i_\lambda i_\mu, \text{ where } \mu>\lambda;$$

$$\text{Average value of } \Sigma P_1^3=\frac{1}{4}(\Sigma i)(\Sigma i-1)\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda)^3 i_\lambda i_\mu, \text{ where } \mu>\lambda.$$

There is no difficulty in continuing the series, which ultimately, of course, involve the numbers of Bernouilli.

It will be noted that the above results enable us to find the average value of

$$\Sigma P_1 P_2.$$

Applied to the assemblage $\alpha^i\beta^j\gamma^k$, we find

$$\text{Average value of } \Sigma P_1^2=\frac{1}{6}(2i+2j+2k-1)(4ik+ij+jk),$$

$$\text{Average value of } \Sigma P_1^3=\frac{1}{4}(i+j+k)(i+j+k-1)(8ik+ij+jk),$$

. . . .

45. The same course of reasoning applies to the minor index

$$R=R_1+R_2+\ldots+R_m,$$

and we thus find that, for the most general assemblage,

$$\Sigma R=\Sigma P;$$

and that the average values of ΣR_1^t and ΣP_1^t are identical. This is somewhat remarkable, because the minor index function Σx^R is not equal in general to the major index function Σx^P; nor is there in general any one-to-one equality between the values of R and the values of P for the same assemblage. The reader may test this statement by looking into the indices of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}$ when $i_1=i_2=i_4=1$ and $i_3=0$. However, there is a one-to-one equality between the numbers P_g and the numbers R_g for the whole of the permutations of the assemblage. Indeed, if this were not the case, we could not have

$$\Sigma P_g^t = \Sigma R_g^t,$$

the summation being for all the permutations, and our results absolutely necessitate this.

46. *General Connection between Σx^P and Partitions.* With the object of investigating the functions Σx^P, Σx^R, I next place them in general connection with the theory of partitions. Consider any permutation

$$\beta\alpha\gamma\gamma\alpha\gamma\beta\alpha\gamma$$

of any assemblage $\alpha^3\beta^2\gamma^4$. Associate with it three rows of numbers

$$\begin{array}{l} a_1\, a_2\, a_3 \\ b_1\, b_2 \\ c_1\, c_2\, c_3\, c_4 \end{array}$$

the numbers in each row being in descending order of magnitude from left to right. I suppose the permutation to correspond to a descending order of magnitude, subject to conditions, of these nine numbers. I regard the permutation as yielding the Diophantine relations

$$\overset{\beta}{b_1} > \overset{\alpha}{a_1} \geq \overset{\gamma}{c_1} \geq \overset{\gamma}{c_2} > \overbrace{\overset{\alpha}{a_2}+1;\quad \overset{\alpha}{a_2}} \geq \overset{\gamma}{c_3} > \overset{\beta}{b_2} > \overset{\alpha}{a_3} \geq \overset{\gamma}{c_4},$$

where it will be noted that in correspondence with the letters α, β, γ, we have the letters a, the letters b and the letters c, each in the *given* descending order. The contacts $\beta\alpha$, $\gamma\beta$ are associated with the symbol $>$; the contact $\gamma\alpha$ also with the symbol $>$; but the letter a, corresponding to α, in this case is increased by unity. So, in general, when the contact is $\alpha_{\mu+\nu}\alpha_\mu$, ν being >0, the symbol is $>$, but the letter corresponding to α_μ is increased by $\nu-1$. When the contact is $\alpha_\mu\alpha_{\mu+\nu}$, where ν is zero or positive, the symbol is $\geq$, and the letter corresponding to $\alpha_{\mu+\nu}$ is not increased or diminished.

Under these circumstances the enumerating generating function of the numbers a, b, c, which satisfy the conditions, is

$$\frac{x^{22}}{(1)(2)\ldots.(9)},$$

where the number which is the exponent of x in the numerator is necessarily the major index of the permutation (compare Art. 6). If we go through a like process for every permutation of the given assemblage and add the generating functions, we find

$$\frac{\Sigma x^P}{(1)(2)\ldots.(9)},$$

Σx^P being the major index function of the given assemblage $\alpha^3\beta^2\gamma^4$.

47. The question now arises: What have we enumerated by this generating function? We clearly have not counted the number of ways of arranging the numbers a, b, c in the three rows so that there may be descending order in each row and also a sum equal to a given number w, for that is what we succeeded in doing by means of the greater index function Σx^p. We have in fact eliminated certain arrangements from those under enumeration. It is easy to see that with reference to the contacts $\alpha\gamma$, $\gamma\alpha$, the former gives

$$\text{Either } a_1 \text{ or } a_2 \text{ or } a_3 \geq \text{either } c_1,\ c_2,\ c_3, \text{ or } c_4,$$

while the latter gives

$$\text{Either } c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } c_4 > \text{either } a_1+1 \text{ or } a_2+1 \text{ or } a_3+1,$$

so that in no circumstances can any number in the third row be *one* greater than any number in the first row. There is no other condition, so that we have enumerated all of the arrangements that were connected with the greater index function *except* those which involve any third row number greater by one than any first row number. Having obtained the precise condition of partition corresponding to the major index function of the assemblage $\alpha^i\beta^j\gamma^k$, we are now in a position to investigate the major index function. Before doing this it is convenient to establish the like conditions that are associated with the major index function of the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$.

48. In correspondence with the contact $\alpha_\mu\alpha_{\mu+\nu}$, we have as a condition of summation

$$\text{Some } \mu\text{-th row number} \geq \text{some } (\mu+\nu)\text{-th row number},$$

and for the contact $\alpha_{\mu+\nu}\alpha_\mu$

$$\text{Some } (\mu+\nu)\text{-th row number} > \text{some } \mu\text{-th row number} + \nu-1;$$

hence under no circumstances can a $(\mu+\nu)$-th row number be greater than a μ-th row number by any of the numbers

$$1, 2, 3, \ldots., \nu-1.$$

In regard to the rows under examination, there are thus $(\nu-1)i_\mu i_{\mu+\nu}$ conditions. Between adjacent rows there are no conditions. Altogether there are

$$\Sigma(\nu-1)\,i_\mu i_{\mu+\nu}$$

conditions, the summation extending to every pair of the numbers $i_1, i_2, \dots, i_s$ whose subscripts differ, at least, by the number 2.

If we now find the function which enumerates the ways in which numbers, of given sum, can be arranged in s rows (containing respectively $i_1, i_2, \dots, i$ numbers, zero being permissible as a number) in descending order in each row so that the above-mentioned $\Sigma(\nu-1)i_\mu i_{\mu+\nu}$ conditions are satisfied, we shall evidently obtain an expression equivalent to

$$\frac{\Sigma x^P}{(1)\,(2)\dots(\Sigma i)},$$

and thus we can arrive at the form of the major index function Σx^P.

The minor index function Σx^R presents no further difficulty, since it was shown in Art. 34 that Σx^R for the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_s^{i_s}$ is identical with Σx^P for the assemblage $\alpha_1^{i_s}\alpha_2^{i_{s-1}}\dots\alpha_s^{i_1}$.

49. *Determination of Σx^P for the Assemblage $\alpha^i\beta^j\gamma^k$.* The associated problem in partitions is merely concerned with the first and third rows of the three rows of numbers. We have thus two rows

$$a_1a_2\dots a_i,$$
$$c_1c_2\dots c_k,$$

and we have to find $\Sigma x^{a_1+a_2+\dots+a_i+c_1+c_2+\dots+c_k}$ subject to the conditions

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_i,$$
$$c_1 \geq c_2 \geq c_3 \geq \dots \geq c_k,$$

with the added set of conditions that no c is to be one greater than any a. Let this function be $\phi_{i,k}(x)$. Then we shall have

$$\frac{\Sigma x^P}{(1)\,(2)\dots(i+j+k)} = \frac{\phi_{i,k}(x)}{(1)\,(2)\dots(j)}.$$

We will first determine $\phi_{1,1}(x)$. We have to find Σx^{a+c} subject to the single condition that c is not to be equal to $a+1$. This is very simple, because we have merely to exclude the terms for which $c=a+1$ from the complete function $\frac{1}{(1)^2}$. Thus

$$\phi_{1,1}(x) = \frac{1}{(1)_2} - \Sigma x^{2a+1} = \frac{1}{(1)^2} - \frac{1}{(2)} = \frac{(4)}{(1)\,(2)^2}.$$

It is, however, more advantageous to proceed by the method that is of general application. For this purpose I construct the function

$$\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)},$$

where the symbol χ denotes that the function is to be expanded in ascending powers of the variable; that, from such expansion, all terms are to be rejected which do not involve either a positive or a negative power of λ; and that then λ is to be put equal to unity. *Ex. gr.*, the general term is

$$\lambda(\lambda x)^a\left(\frac{x}{\lambda}\right)^c \quad \text{or} \quad \lambda^{a+1-c}x^{a+c};$$

so that the symbol χ rejects terms for which $c=a+1$, and thus the final function is Σx^{a+c}, the sum being for all positive integers a, c except those for which $c=a+1$. This is the required condition, so that

$$\phi_{1,1}(x)=\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}.$$

Observe that the χ function is *necessarily* unaltered when we write $\frac{1}{\lambda}$ for λ, although its form is changed. We have the identity

$$\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}=\frac{\lambda}{(2)\,(1-\lambda x)}+\frac{\frac{x^2}{\lambda}}{(2)\left(1-\frac{x}{\lambda}\right)}+\frac{x}{(2)};$$

so that

$$\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}=\frac{1}{(1)\,(2)}+\frac{x^2}{(1)\,(2)}=\frac{(4)}{(1)\,(2)^2};$$

leading of course to the result that, for the assemblage $\alpha\beta^j\gamma$, Σx^P has the expression

$$\frac{(1)\,(2)\,\ldots\ldots\,(\mathrm{j}+2)}{(1)\,(2)\,\ldots\ldots\,(\mathrm{j})}\,\frac{(4)}{(1)\,(2)^2} \quad \text{or} \quad \frac{(\mathrm{j}+1)\,(\mathrm{i}+2)\cdot(4)}{(1)\,(2)\cdot(2)}.$$

50. To determine $\phi_{2,1}(x)$ I form a function

$$\chi\cdot\Omega\cdot\frac{\lambda\mu}{(1-A\lambda x)\left(1-\frac{\mu x}{A}\right)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The general term in the expansion of the function is

$$\lambda\mu(A\lambda x)^{a_1}\left(\frac{\mu x}{A}\right)^{a_2}\left(\frac{x}{\lambda\mu}\right)^{c_1} \quad \text{or} \quad A^{a_1-a_2}\lambda^{a_1+1-c_1}\mu^{a_2+1-c_1}x^{a_1+a_2+c_1}.$$

If, therefore, in the expansion, the exponent of A is zero or positive, the condition $a_1\geq a_2$ is satisfied. If also the exponents of λ and μ are neither of them zero, the conditions $a_1+1\neq c_1$, $a_2+1\neq c_1$ are satisfied. The symbol Ω is associated with the auxiliary coefficient A in such wise that its effect upon the expanded function is to abolish all terms which involve A to a negative power. The symbol χ is associated with the auxiliaries λ and μ in such wise that it abolishes from the expansion all terms which do not involve λ and μ (each of them) positively or negatively. Hence, from our definitions

$$\phi_{2,1}(x)=\chi\cdot\Omega\cdot\frac{\lambda\mu}{(1-A\lambda x)\left(1-\frac{\mu x}{A}\right)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The effect of Ω can be seen with little trouble, so that we may write

$$\phi_{2,1}(x)=\chi\cdot\frac{\lambda\mu}{(1-\lambda x)(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The reader will have little difficulty in proving that, if σ be any auxiliary or combination of auxiliaries,

$$\chi\frac{\sigma}{(1-\sigma x^s)\left(1-\frac{x^t}{\sigma}\right)}=\frac{\sigma}{(\mathrm{s}+\mathrm{t})\,(1-\sigma x^s)}+\frac{\frac{1}{\sigma}x^{2t}}{(\mathrm{s}+\mathrm{t})\left(1-\frac{x^t}{\sigma}\right)},\quad \sigma=1;$$

$$\chi\frac{\frac{1}{\sigma}}{(1-\sigma x^s)\left(1-\frac{x^t}{\sigma}\right)}=\frac{\sigma x^{2s}}{(\mathrm{s}+\mathrm{t})\,(1-\sigma x^s)}+\frac{\frac{1}{\sigma}}{(\mathrm{s}+\mathrm{t})\left(1-\frac{x^t}{\sigma}\right)},\quad \sigma=1.$$

Applying the first of these results we can eliminate the auxiliary μ. We have

$$\phi_{2,1}(x)=\chi\frac{1}{1-\lambda x}\left\{\frac{\lambda}{(3)\,(1-\lambda x^2)}+\frac{\frac{1}{\lambda}x^2}{(3)\left(1-\frac{x}{\lambda}\right)}\right\}$$

$$=\frac{1}{(1)\,(2)\,(3)}+\frac{x^2}{(3)}\chi\frac{\frac{1}{\lambda}}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)};$$

and, since in the χ function we can write $\frac{1}{\lambda}$ for λ and $\phi_{2,0}=\frac{1}{(1)\,(2)}$, we find

$$\phi_{2,1}(x)=\frac{1}{(3)}\{\phi_{2,0}(x)+x^2\phi_{1,1}(x)\},$$

a suggestive formula of reduction.

51. In a precisely similar manner we find

$$\phi_{3,1}(x)=\chi\cdot\Omega\cdot\frac{\lambda\mu\nu}{(1-A_1\lambda x)\left(1-\frac{A_2\mu x}{A_1}\right)\left(1-\frac{\nu x}{A_2}\right)\cdot\left(1-\frac{x}{\lambda\mu\nu}\right)},$$

where Ω abolishes the terms which involve negative powers of A_1 and A_2, and χ acts in the usual manner on the auxiliaries λ, μ, ν.

We find at once

$$\phi_{3,1}(x)=\chi\frac{\lambda\mu\nu}{(1-\lambda x)(1-\lambda\mu x^2)(1-\lambda\mu\nu x^3)\cdot\left(1-\frac{x}{\lambda\mu\nu}\right)};$$

and now, eliminating ν, we find

$$\phi_{3,1}(x)=\chi\frac{1}{(1-\lambda x)(1-\lambda\mu x^2)}\cdot\left\{\frac{\lambda\mu}{(4)(1-\lambda\mu x^3)}+\frac{\frac{1}{\lambda\mu}x^2}{(4)\left(1-\frac{x}{\lambda\mu}\right)}\right\},$$

$$=\frac{1}{(4)}\phi_{3,0}(x)+\frac{x^2}{(4)}\chi\frac{\frac{1}{\lambda\mu}}{(1-\lambda x)(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}.$$

Now

$$\chi\frac{\frac{1}{\lambda\mu}}{(1-\lambda x)(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}$$

is not of the same form as the expression for $\phi_{2,1}(x)$; but it is equivalent to it, because, from the fact that $\phi_{i,k}(x)=\phi_{k,i}(x)$, established above, if we proceed to form $\phi_{1,2}(x)$ we find

$$\phi_{1,2}(x)=\chi\frac{\lambda\mu}{(1-\lambda\mu x)\left(1-\frac{x}{\lambda}\right)\left(1-\frac{x^2}{\lambda\mu}\right)},$$

and herein we may substitute $\frac{1}{\lambda}$ for λ and $\frac{1}{\mu}$ for μ.

Hence

$$\phi_{3,1}(x)=\frac{1}{(4)}\phi_{3,0}(x)+\frac{x^2}{(4)}\phi_{2,1}(x).$$

52. Examination of the foregoing shows that

$$\phi_{s,1}(x)=\frac{1}{(s+1)}\phi_{s,0}(x)+\frac{x^2}{(s+1)}\phi_{s-1,1}(x),$$

and assuming that

$$\phi_{s-1,1}(x)=\frac{(2s)}{(1)(2)^2(3)\ldots.(s)}$$

we find

$$\phi_{s,1}(x) = \frac{1}{(1)(2)\ldots\ldots(s+1)} + \frac{x^2(1-x^{2s})}{(1)(2)^2(3)\ldots\ldots(s+1)} = \frac{(2s+2)}{(1)(2)^2(3)\ldots\ldots(s+1)},$$

verifying the assumption.

Hence Σx^P for the assemblage $\alpha^i\beta^j\gamma$ is

$$\frac{(1)(2)\ldots\ldots(i+j+1)}{(1)(2)\ldots\ldots(j)\cdot(1)(2)\ldots\ldots(i+1)} \cdot \frac{(2)(4)\ldots\ldots(2i+2)}{(2)(4)\ldots\ldots(2i)\cdot(2)}.$$

53. We now examine the series $\phi_{2,2}(x)$, $\phi_{3,2}(x)$, ... It will be agreed that

$$\phi_{2,2}(x) = \chi\cdot\Omega\frac{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}{(1-A\lambda_{11}\lambda_{12}x)\left(1-\frac{\lambda_{21}\lambda_{22}}{A}x\right)\cdot\left(1-\frac{B}{\lambda_{11}\lambda_{21}}x\right)\left(1-\frac{x}{B\lambda_{12}\lambda_{22}}\right)},$$

where Ω has reference to A and B, and χ to the auxiliaries λ_{11}, λ_{12}, λ_{21}, λ_{22}. We immediately eliminate A and B and find

$$\phi_{2,2}(x) = \chi\cdot\frac{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}{(1-\lambda_{11}\lambda_{12}x)(1-\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}x^2)\cdot\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)\left(1-\frac{x^2}{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}\right)}.$$

Eliminating λ_{22} we find

$$\phi_{2,2}(x) = \chi\cdot\frac{1}{(1-\lambda_{11}\lambda_{12}x)\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)(4)}\left\{\frac{\lambda_{11}\lambda_{12}\lambda_{21}}{1-\lambda_{11}\lambda_{12}\lambda_{21}x^2} + \frac{\frac{1}{\lambda_{11}\lambda_{12}\lambda_{21}}x^4}{1-\frac{x^2}{\lambda_{11}\lambda_{12}\lambda_{21}}}\right\}.$$

$\phi_{2,2}(x)$ is now the sum of two χ functions; in the first of these we may put $\lambda_{12}=1$, since λ_{12} occurs to a positive power in every term of the expansion, and similarly we may put $\lambda_{21}=1$ in the second function, for it occurs to a negative power in every term of the expansion. Hence

$$\phi_{2,2}(x) = \frac{1}{(4)}\chi\cdot\frac{\lambda_{11}\lambda_{21}}{(1-\lambda_{11}x)(1-\lambda_{11}\lambda_{21}x^2)\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)}$$

$$+\frac{x^4}{(4)}\chi\cdot\frac{\frac{1}{\lambda_{11}\lambda_{12}}}{(1-\lambda_{11}\lambda_{12}x)\left(1-\frac{x}{\lambda_{11}}\right)\left(1-\frac{x^2}{\lambda_{11}\lambda_{12}}\right)},$$

and from previous results we may write

$$\phi_{2,2}(x) = \frac{1}{(4)}\phi_{2,1}(x) + \frac{x^4}{(4)}\phi_{1,2}(x).$$

54. Similarly we find

$$\phi_{3,2}(x) = \frac{1}{(5)}\phi_{3,1}(x) + \frac{x^4}{(5)}\phi_{2,2}(x);$$

and in general

$$\phi_{i,2}(x) = \frac{1}{(i+2)}\phi_{i,1}(x) + \frac{x^4}{(i+2)}\phi_{i-1,2}(x).$$

We know the value of $\phi_{1,2}(x)$, and assuming that

$$\phi_{i-1,2}(x) = \frac{1}{(1)\,(2)\ldots\ldots(i+1)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+2)}{(2)\,(4)\ldots\ldots(2i-2)\cdot(2)\,(4)},$$

we find

$$\begin{aligned}\phi_{i,2}(x) &= \frac{(2i+2)}{(1)\,(2)^2(3)\,(4)\ldots\ldots(i+2)} + \frac{x^4}{(1)\,(2)\ldots\ldots(i+2)}\cdot\frac{(2i)\,(2i+2)}{(2)\,(4)},\\ &= \frac{(2i+2)}{(1)\,(2)^2(3)\,(4)^2(5)\ldots\ldots(i+2)}\{1-x^4+x^4-x^{2i+4}\},\\ &= \frac{(2i+2)\,(2i+4)}{(1)\,(2)^2(3)\,(4)^2(5)\ldots\ldots(i+2)},\\ &= \frac{1}{(1)\,(2)\ldots\ldots(i+2)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+4)}{(2)\,(4)\ldots\ldots(2i)\cdot(2)\,(4)};\end{aligned}$$

establishing the assumed law.

Hence Σx^P for the assemblage $\alpha^i\beta^j\gamma^2$ is

$$\frac{(1)\,(2)\ldots\ldots(i+j+2)}{(1)\,(2)\ldots\ldots(j)\cdot(1)\,(2)\ldots\ldots(i+2)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+4)}{(2)\,(4)\ldots\ldots(2i)\cdot(2)\,(4)}.$$

55. On precisely the same reasoning we find

$$\phi_{i,k}(x) = \frac{1}{(i+k)}\phi_{i,k-1}(x) + \frac{x^{2k}}{(i+k)}\phi_{i-1,k}(x);$$

and assuming that

$$\phi_{i,k-1}(x) = \frac{1}{(1)\,(2)\ldots\ldots(i+k-1)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+2k-2)}{(2)\,(4)\ldots\ldots(2i)\cdot(2)\,(4)\ldots\ldots(2k-2)},$$

$$\phi_{i-1,k}(x) = \frac{1}{(1)\,(2)\ldots\ldots(i+k-1)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+2k-2)}{(2)\,(4)\ldots\ldots(2i-2)\cdot(2)\,(4)\ldots\ldots(2k)},$$

we find

$$\phi_{i,k}(x) = \frac{1}{(1)\,(2)\ldots\ldots(i+k)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+2k-2)}{(2)\,(4)\ldots\ldots(2i)\cdot(2)\,(4)\ldots\ldots(2k)}\{1-x^{2k} + x^{2k}(1-x^{2i})\},$$

or

$$\phi_{i,k}(x) = \frac{1}{(1)\,(2)\ldots\ldots(i+k)}\cdot\frac{(2)\,(4)\ldots\ldots(2i+2k)}{(2)\,(4)\ldots\ldots(2i)\cdot(2)\,(4)\ldots\ldots(2k)},$$

the final result.

Hence Σx^P for the assemblage $\alpha^i\beta^j\gamma^k$ is

$$\frac{(1)(2)\ldots.(i+j+k)}{(1)(2)\ldots.(j)\cdot(1)(2)\ldots.(i+k)}\cdot\frac{(2)(4)\ldots.(2i+2k)}{(2)(4)\ldots.(2i)\cdot(2)(4)\ldots.(2k)},$$

or

$$F_{j,i+k}(x)\cdot F_{i,k}(x^2),$$

as previously conjectured.

56. *The Expression of Σx^P for the Assemblage $\alpha^i\beta^j\gamma^k\delta^l$.* The actual determination of this expression is reserved for a future occasion, but a few remarks upon the problem will not be out of place. For the assemblage $\alpha^i\beta^j\gamma^k$ the second row of numbers was not involved in the partition question, so that we had only two rows to deal with. In the present instance we have all four rows essentially involved in the partition question. We may write

$$\frac{\Sigma x^P}{(1)(2)\ldots.(i+j+k+l)}=\phi_{i,j,k,l}(x),$$

and the question in partition is the determination of $\phi_{i,j,k,l}(x)$ by considering the four rows of numbers

$$\begin{gathered}a_1\,a_2\,a_3\ldots.\,a_i,\\ b_1\,b_2\,b_3\ldots.\,b_j,\\ c_1\,c_2\,c_3\ldots.\,c_k,\\ d_1\,d_2\,d_3\ldots.\,d_l.\end{gathered}$$

We have to find $\Sigma x^{\Sigma a+\Sigma b+\Sigma c+\Sigma d}$ from the conditions

$$\begin{gathered}a_1\geq a_2\geq a_3\geq\ldots.\geq a_i,\\ b_1\geq b_2\geq b_3\geq\ldots.\geq b_j,\\ c_1\geq c_2\geq c_3\geq\ldots.\geq c_k,\\ d_1\geq d_2\geq d_3\geq\ldots.\geq d_l,\end{gathered}$$

with the added conditions:

(i) No c is to be one greater than any a.
(ii) No d is to be one greater than any b.
(iii) No d is to be one greater or two greater than any a.

There is no difficulty in writing down the crude generating function involving the symbols χ and Ω and the associated auxiliary coefficients. There is, however, considerable difficulty in handling the function in question so as to eliminate all the auxiliaries and arrive at the desired formula.

The function Σx^P or $(1)(2)\ldots.(i+j+k+l)\,\phi_{i,j,k,l}(x)$ has the following properties:

(i) For $x=1$ it must reduce to the number which enumerates the permutation of the assemblage $\alpha^i\beta^j\gamma^k\delta^l$.

(ii) If x be put equal to unity after differentiation with regard to x, we must obtain the value of ΣP that has been set forth in Art. 43.

(iii) For $j=k=0$ it must reduce to

$$F_{i,l}(x^3).$$

(iv) For $l=0$ and $i=0$ it must reduce to

$$F_{j,i+k}(x)\cdot F_{i,k}(x^2) \quad \text{and} \quad F_{k,j+l}(x)\cdot F_{j,l}(x^2)$$

respectively.

If we put $k=0$ we must get the expression of Σx^P for the assemblage $\alpha^i\beta^j\delta^l$; if we put $j=0$ we must get the expression of Σx^P for the assemblage $\alpha^i\gamma^k\delta^l$; and it would seem to be imperative to investigate these expressions in the first instance. In fact, when we come to the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$, we may put any $s-3$ of the repetitional exponents equal to zero, and we would thus realize the expression of Σx^P for the assemblage

$$\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}\alpha_{s_3}^{i_{s_3}},$$

where s_1, s_2, s_3 are any three different numbers in ascending order of magnitude. It would thus appear to be imperative ultimately to investigate this case also.

It is of course obvious that Σx^P for the assemblage $\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}$ is

$$F_{i_{s_1}}, F_{i_{s_2}}(x^{s_2-s_1}),$$

and we thus obtain three indications to help us in the investigation of the assemblage

$$\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}\alpha_{s_3}^{i_s}.$$

To these we may add the known results of putting $x=1$ before and after differentiation.

57. *Crude Expression for Σx^P for the Assemblage $\alpha\beta\gamma\delta$.* The actual expression of Σx^P can be obtained from the permutation set forth in Art. 32. It is

$$1+x+2x^2+3x^3+4x^4+2x^5+5x^6+x^7+3x^8+x^9+x^{10}.$$

Its unsymmetrical character will be noted. It shows that we can not expect the expression of Σx^P for the assemblage $\alpha^i\beta^j\gamma^k\delta^l$ to be in the form of a single algebraic fraction involving factors all of the form (s).

For the assemblage $\alpha\beta\gamma\delta$ I construct the function

$$\chi\cdot\frac{\lambda_{13}\lambda_{14}\lambda_{24}\mu_{14}^2}{(1-\lambda_{13}\lambda_{14}\mu_{14}x)(1-\lambda_{24}x)\left(1-\frac{x}{\lambda_{13}}\right)\left(1-\frac{x}{\lambda_{14}\lambda_{24}\mu_{14}}\right)}$$

and note that the general term in the expansion is

$$\lambda_{13}\lambda_{14}\lambda_{24}\mu_{14}^2(\lambda_{13}\lambda_{14}\mu_{14}x)^a(\lambda_{24}x)^b\left(\frac{x}{\lambda_{13}}\right)^c\left(\frac{x}{\lambda_{14}\lambda_{24}\mu_{14}}\right)^d$$

or

$$\lambda_{13}^{a+1-c}\lambda_{14}^{a+1-d}\lambda_{24}^{b+1-d}\mu_{14}^{a+2-d}x^{a+b+c+d}.$$

The operation of χ excludes zero powers of each of the four auxiliaries, so that the required conditions are satisfied, and the constructed function is therefore the crude expression of $\phi_{1,1,1,1}(x)$. Then

$$\Sigma x^P=(1)(2)(3)(4)\phi_{1,1,1,1}(x).$$

I have actually verified that

$$\phi_{1,1,1,1}(x)=\frac{1+x+2x^2+3x^3+4x^4+2x^5+5x^6+x^7+3x^8+x^9+x^{10}}{(1)(2)(3)(4)},$$

but as I have not yet succeeded in carrying out the reduction in a satisfactory manner, so as to show how the general case should be handled, I withhold the algebra.

London, England, *October* 21, 1912.